Effects of disorder on modulated quantum Hall systems

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Abstract

The Hall conductivity and the localization length are calculated for weakly modulated two-dimensional systems within the lowest Landau level approximation. We find that the fractal character of the Hofstadter butterfly is reflected on the coincidence in the localization and the Hall conductivity among a series of fluxes \( \phi + 2n \) with integers \( n \).

Key words: quantum Hall effect, the Hofstadter butterfly, Anderson localization

1. Introduction

A two dimensional (2D) electron system with a 2D periodic potential is known to exhibit an intricate energy spectrum in a strong magnetic field, which is called the Hofstadter butterfly. The interplay of the Landau quantization and Bragg’s reflection yields a fractal-like series of the energy gaps which depends sensitively on the number of magnetic flux quanta penetrating a unit cell [1]. The Hall conductivity is quantized when the Fermi energy lies in each of the gaps, varying in non-trivial manner from one gap to another [2]. Recently the evidence of the fractal energy spectrum was found in the superlattice patterned on GaAs/AlGaAs heterostructures [3,4], where the Hall conductivity changes nonmonotonically as a function of the field.

By analogy with the unmodulated 2D systems, we expect that in an enough large system with modulations, the disorder localized almost all the states in Hofstadter’s butterfly and the Hall conductivity is quantized around every minigap. As a result \( \sigma_{xy}(E) \) would turn into a series of the Hall plateaus, which are separated by the extended states existing at certain energies. Several authors studied the evolution of the extended states in the Hofstadter butterfly as a function of the disorder strength for several flux states [6,7].

In the previous work, the Hall conductivity was calculated in weakly modulated 2D systems with finite sizes [5], where it was shown that \( \sigma_{xy} \) becomes independent of the system size at \( \sigma_{xy} = 1/2 \) (in units of \( -e^2/h \)), and those fixed points can be identified as the critical energies in an infinite system. While there we concentrated on a few limited fluxes, it is still a remaining question how the quantum Hall effect looks like in general fluxes over the whole region of the Hofstadter butterfly. Especially it is intriguing to consider how the self-similar structure, one of the most fascinating property in this system, is reflected on the localization. In this paper, we calculate the Hall conductivity and the localization length in disordered 2D periodic systems for a self-similar series of fluxes, and find that the localization lengths agree among them when the integers of \( \sigma_{xy} \) assigned to subbands are identical.

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2. Formulation

We consider the one-particle Hamiltonian for a two-dimensional system in a strong magnetic field, which is given by

\[ H = \frac{1}{2m}(p + eA)^2 + V_p + V_d, \]

where \( V_d \) is a disorder potential and \( V_p \) has a two-dimensionally periodic potential with a square form,

\[ V_p = V \left( \cos \frac{2\pi}{a} x + \cos \frac{2\pi}{a} y \right). \]

The band structure in an ideal system (\( V_d = 0 \)) is then characterized by a parameter \( \phi = B a^2 / (\hbar e) \), a number of magnetic flux quanta penetrating unit cell [1]. For the disorder potential \( V_d \), we take randomly distributed delta-potentials \( \pm u_0 \) with the equal amounts of the positive and negative scatterers, where the total number of the scatterers in a unit area \( n_i \) is given by \( 2\pi l^2 n_i = 4 \), with the magnetic length \( l = \sqrt{\hbar / (eB)} \). The energy scale for the disorder is given by \( \Gamma = \{ 4n_i u_0^2 / (2\pi l^2) \}^{1/2} \), which represents the broadening width of the Landau level in a unmodulated system [8]. We consider only the lowest Landau level, assuming that the magnetic field is strong enough and the mixing of the Landau levels is neglected. We estimate the localization length by the Thouless number method [9], and calculate the Hall conductivity using the Kubo formula for zero temperature,

\[ \sigma_{xy} = \frac{\hbar e^2}{4L^2} \sum_{\nu\alpha<\nu\beta \neq \alpha} (\epsilon_{\nu} | \langle \nu | \beta \rangle \rangle | \nu \rangle) - (\epsilon_{\alpha} | \langle \alpha | \beta \rangle \rangle | \alpha \rangle) / (\epsilon_{\alpha} - \epsilon_{\beta})^2, \]

where \( \epsilon_{\nu} \) is the energy of the eigenstate \( | \alpha \rangle \), \( E_F \) the Fermi energy, \( v_i \) the velocity operator, and \( L \) is the system size.

3. Results

Fig. 1 shows the energy spectrum in the lowest Landau level in an ideal system (\( V_d = 0 \)) plotted against \( \phi \), where the integers indicate the quantized Hall conductivity \( \sigma_{xy} \) when \( E_F \) lies in each of the gaps. While we have the similar subband structure everywhere, the localization properties do not generally agree among them unless the Hall conductivities \( \sigma_{xy} \) assigned to the gaps are identical. For example \( \phi = 3/2 \) and 3 have the similar spectra composed of three subbands, but exhibit the different behaviors in the localization: the upper and the lower subbands in \( \phi = 3 \) carry zero Hall conductivity, all the subbands in \( \phi = 3/2 \) carry non-zero \( \sigma_{xy} \) and they each have the extended levels [5].

On the other hand, we have a self-similar series with the identical distribution of the Hall conductivity as indicated in Fig. 1, where the entire structure is equivalent to the shaded region. Specifically, the total \( p \) subbands at \( \phi = p/q \) (\( p, q \) co-prime integers) correspond to the \( p \) subbands in the middle at \( \phi + 2 \). There we can show that the number of states in one subband is the same between \( \phi \) and \( \phi + 2 \) in a fixed lattice constant \( a \), so that the Hall conductivities in corresponding gaps become identical as follows from the Streda formula [10],

\[ \sigma_{xy} = -\frac{\epsilon}{\partial B}, \]

where \( \rho \) is the number of states below the Fermi energy per unit area and \( B \) is the magnetic field. We have the similar relation among the fluxes \( \phi + 2n \) for arbitrary integer \( n \), so that the structure repeats infinitely.
In Fig. 3, we compare the disorder effects on the density of states, the Hall conductivity and the localization length, where we chose the disorder strengths appropriately so that the ratio of the DOS broadening $\gamma$ to $W$ is identically 0.18. $\gamma$ is defined as

$$\gamma = \frac{2\pi n_0 \sigma_{xy}^2}{W},$$

which is approximately related to the life time of a Bloch state $\tau$ via $\gamma = \hbar/\tau$. This quantity is different from $\Gamma$, or the broadening of the Landau level in absence of the modulation. The result shows that the broadened densities of states are almost identical, while the width of the center subband slightly varies due to the difference in the ideal spectrum. More remarkably we have quantitative agreement also in the localization length $L_{loc}$ in units of the lattice constant $a$, suggesting that the localization in the Hofstadter butterfly is uniquely determined by the band structure and the Hall conductivity assigned to the subbands. The Hall conductivity $\sigma_{xy}$ for the disordered system, shown only for a finite system size $L = 16a$, also agrees quite well, except for the a small deviation in the middle band coming from the ideal spectrum. As the system becomes larger, $\sigma_{xy}(E)$ gradually approaches 1 in the area $\sigma_{xy} > \frac{1}{2}$, and 0 in $\sigma_{xy} < \frac{1}{2}$, and finally reaches the quantized value when the system size exceeds the localization length $L_{loc}$ [5]. Now we have the agreement in $L_{loc}(E)$ between two fluxes, so that we have the almost identical curve of $\sigma_{xy}(E)$ at the same system size $L = 16a$.

The critical energies for the extended levels in an infinite system can be identified as the points of $\sigma_{xy} = \frac{1}{2}$ in a finite system [5]. We can show that three extended levels in $\phi = \frac{3}{2}$ and $\frac{7}{2}$ evolve in a parallel fashion with the disorder strength $\gamma/W$, where they come closer as $\gamma/W$ becomes larger and combine into one at $\gamma/W \approx O(1)$. For too strong disorder to destroy the whole subband structure, or $\gamma/W \gg 1$, the property crosses over to that of the unmodulated case where $L_{loc}$ scales with the magnetic length $l$ depending on the field amplitude $B$ [9].

The similar subband structures with the identical Hall conductivity can be found as different hierarchies in the Hofstadter butterfly. The tiling with the smallest unit that we have found is schematically shown in Fig. 4, where we can see that the structure between $\phi = 0$ and $1$ repeats over the entire region. We expect that

![Fig. 2. Density of states and the Hall conductivity for ideal systems with $\phi = \frac{3}{2}$ (solid lines) and $\frac{7}{2}$ (dashed).](image)

![Fig. 3. (a) Density of states, (b) the Hall conductivity ($L = 16a$) and (c) the inverse localization length calculated for disordered systems with $\phi = \frac{3}{2}$ (solid lines) and $\frac{7}{2}$ (dashed).](image)
the localization over those structures becomes similar, while the deformation of the unit should make a quantitative difference.

**Conclusion**

We have studied the quantum Hall effect in disordered 2D electron systems with 2D periodic potential at several fluxes. We have found that a self-similar subband structure with an identical sequence of the Hall conductivity emerges in a series of the magnetic fluxes $\phi + 2n$, and that the localization lengths agree quantitatively among the corresponding fluxes.

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**References**