A brief review is given on electronic and transport properties of a two-dimensional honeycomb lattice from a theoretical point of view. The topics include the symmetry crossover among symplectic, unitary, and orthogonal due to the presence of special time reversal symmetry and the topological anomaly giving rise to various zero-mode anomalies and peculiar magnetotransport.

Keywords: graphene; neutrino; topological anomaly; anti-localization; quantum Hall effect; zero-mode anomaly.

1. Introduction

Quite recently, an atomically thin graphene, or a single layer graphite, was fabricated, and the magnetotransport was measured and the integer quantum Hall effect was observed. In an effective-mass approximation, an electron in a graphite monolayer is described by Weyl’s equation for a massless neutrino. Transport properties in such an exotic electronic structure are quite intriguing, and the conductivity with/without a magnetic field including the Hall effect, quantum corrections to the conductivity, the dynamical transport, and effects of screening and impurity scattering were investigated theoretically. The results show that the two-dimensional graphite exhibits various characteristic behaviors different from conventional two-dimensional systems. The purpose of this paper is to give a brief review on such electronic and transport properties from a theoretical point of view.

2. Neutrino Description

The structure of 2D graphite sheet is shown in Fig. 1. A unit cell contains two carbon atoms which are denoted by A and B. We have the primitive translation vectors \( \mathbf{a} = a(1, 0) \) and \( \mathbf{b} = a(-1/2, \sqrt{3}/2) \), and the vectors connecting between nearest neighbor carbon atoms \( \mathbf{t}_1 = a(0, 1/\sqrt{3}) \), \( \mathbf{t}_2 = a(-1/2, -1/2\sqrt{3}) \), and \( \mathbf{t}_3 = a(1/2, -1/2\sqrt{3}) \), where the lattice constant is given by \( a = 2.46 \) Å. The first Brillouin zone is given by a hexagon with two corner points K and K'. The corresponding
wave vectors are given by $\mathbf{K} = (2\pi/a)(1/3, 1/\sqrt{3})$ and $\mathbf{K}' = (2\pi/a)(2/3, 0)$ for K and K’ points, respectively. In a nearest-neighbor tight-binding model, the energy bands are given by

$$\varepsilon_{\pm}(\mathbf{k}) = \pm \gamma \left| \sum_{l=1}^{3} \exp(i\mathbf{k} \cdot \mathbf{\tau}_l) \right|.$$

The band structure is shown in Fig. 2. Near the K and K’ point, we have $\varepsilon_{\pm}(\mathbf{k} + \mathbf{K}) = \varepsilon_{\pm}(\mathbf{k} + \mathbf{K}') = \pm \gamma |\mathbf{k}|$ with $\gamma = \sqrt{3a\gamma_0}/2$.

Essential and important features of electronic states become transparent when we use a $\mathbf{k} \cdot \mathbf{p}$ scheme in describing states in the vicinity of K and K’ points. The effective-mass equation for the K point is given by

$$\mathcal{H}_0 \mathbf{F}(\mathbf{r}) = \varepsilon \mathbf{F}(\mathbf{r}), \quad \mathcal{H}_0 = \gamma (\hat{\sigma} \cdot \hat{\mathbf{k}}), \quad \mathbf{F}(\mathbf{r}) = \begin{pmatrix} F_A(\mathbf{r}) \\ F_B(\mathbf{r}) \end{pmatrix},$$

where $\hat{\sigma} = (\sigma_x, \sigma_y)$ is the Pauli spin matrix, $\gamma$ is a band parameter, $\hat{\mathbf{k}} = -i\nabla$, and $F_A$ and $F_B$ represent the amplitude at two carbon sites A and B, respectively. The above equation is same as Weyl’s equation for a neutrino with vanishing rest mass and constant velocity independent of the wave vector. The velocity is given by $|\mathbf{v}| = \gamma/h$. For the K’ point the Schrödinger equation is given by Eq. (2) in which $\hat{\sigma}$ is replaced by its complex conjugate $\hat{\sigma}^*$. The density of states becomes $D(\varepsilon) = |\varepsilon|/2\pi\gamma^2$. Figure 3 shows the energy dispersion and the density of states.

### 3. Symmetry Crossover

The Weyl equation (2) is invariant under a special time-reversal operation $S$,

$$\mathbf{F}^S = K \mathbf{F}^*,$$

where $\mathbf{F}^*$ represents the complex conjugate of the wave function $\mathbf{F}$ and $K$ is an antisymmetric unitary matrix $K = -i\sigma_y$ satisfying $K^2 = -1$. The corresponding
operation for an operator $P$ is given by $P^S = K^t PK^{-1}$, where $^tP$ stands for the transpose of $P$. This corresponds to the time reversal in systems with spin-orbit interaction and leads to

$$F^S \equiv (F^S)^S = -F.$$  \hspace{1cm} (4)

The system belongs to the symplectic universality class when only $S$ constitutes a relevant symmetry.\(^\dagger\)

This symmetry prevails even in the presence of impurities unless their potential range is smaller than the lattice constant $a$. In fact, for such scatterers, the effective potential is the same for the A and B sites and does not cause any mixing between the K and K’ points. In this case a quantum correction or a weak-localization correction to the Boltzmann conductivity becomes positive and diverges logarithmically.\(^\dagger\)

This so-called anti-localization behavior is the same as that appears in systems with strong spin-orbit interaction.

The operation $S$ is not the real time-reversal in a two-dimensional graphite. Actually, the Bloch functions at the K and K’ points are mutually complex conjugate and therefore, the K point is converted into the K’ point and the K’ point into the K point under the real time reversal. In the present $k \cdot p$ scheme, this operation $T$ is expressed as

$$F^T_K = e^{-i\psi} \sigma_z F^\ast_{K'}, \quad F^T_{K'} = e^{-i\psi} \sigma_z F^\ast_K,$$  \hspace{1cm} (5)

where $\psi$ is an arbitrary phase factor and $F_K$ and $F_{K'}$ are the wave functions at the K and K’ points, respectively. This immediately gives

$$F^{T^2}_K \equiv (F^T_K)^T = F_K, \quad F^{T^2}_{K'} \equiv (F^T_{K'})^T = F_{K'},$$  \hspace{1cm} (6)

characteristic of the conventional orthogonal symmetry. When we can neglect mixing between the K and K’ points and confine ourselves to states in each valley, however, the $T$ symmetry is irrelevant and the special $S$ symmetry becomes relevant.

In the presence of short-range scatterers causing mixing between K and K’ points, the $S$ symmetry is violated, but the $T$ symmetry prevails. As a result the
system now belongs to the orthogonal class.\cite{11,18} In this case, quantum corrections become negative and the conductivity now shows a weak-localization behavior. A crossover between the anti-localization and weak localization behavior by the presence of short-range scatterers has been demonstrated.\cite{11}

The actual equi-energy line deviates from the circle and has trigonal warping when the energy becomes nonzero. This effect can be included by a higher order $k \cdot p$ term. For the K point it is given by

$$\mathcal{H}_1 = \gamma \left( \begin{array}{cc} 0 & h_1(\hat{k}_x, \hat{k}_y) \\ h_1(\hat{k}_x, \hat{k}_y)^\dagger & 0 \end{array} \right), \quad h_1(\hat{k}_x, \hat{k}_y) = \beta \frac{a}{4\sqrt{3}} (\hat{k}_x + i\hat{k}_y)^2,$$

(7)

where $\beta$ is a constant of the order of unity. This higher order term gives rise to a trigonal warping of the dispersion. Its strength is characterized by the dimensionless parameter $\beta a/L$. This expression with $\beta = 1$ has been derived from a nearest-neighbor tight-binding model,\cite{19} but is expected to be much more general if we regard $\beta$ as an adjustable parameter. In the presence of $\mathcal{H}_1$, the special time reversal symmetry is destroyed because $\mathcal{H}_1^S = -\mathcal{H}_1$.\cite{20} As a result the system now belongs to the unitary class. It should be noted also that the presence of lattice strain gives rise to an effective vector potential and therefore destroys the $S$ symmetry.\cite{21}

The symmetry plays important roles in carbon nanotubes.\cite{22} Figure 4 shows the energy bands in a metallic nanotube with circumference $L$, where the wave vector along the circumference is quantized into $(2\pi/L)j$ with integer $j$. In the presence of the $S$ symmetry, the reflection coefficients satisfy the symmetry relation $r_{\alpha\beta} = -r_{\beta\alpha}$, where $\alpha$ and $\beta$ denote in-coming channels and $\bar{\alpha}$ and $\bar{\beta}$ their time reversal or out-going channels. Metallic nanotubes satisfy this symmetry and possess an odd number of current carrying channels for each of the K and K’ points. The determinant of an antisymmetric matrix with odd dimension vanishes identically. Because the reflection-coefficient matrix possesses this property, there is a perfectly conducting channel which transmits through the system without being scattered back.\cite{23} When only a single conducting channel is present, there is no backward scattering and the conductance is given by an ideal value $2e^2/\pi\hbar$.\cite{24,25}

An interesting feature of nanotubes is the Aharonov-Bohm (AB) effect on the band structure.\cite{16,26} Recently, the splitting of optical absorption and emission peaks due to flux was observed.\cite{27,28,29} In the presence of a magnetic flux the time-reversal symmetry is absent and therefore the perfectly conducting channel is destroyed. Explicit numerical calculations\cite{30} showed that the perfect channel is quite fragile and disappears for flux less than 1 % of the flux quantum $\phi_0 = \hbar c/e$, while the absence of backscattering in linear bands is quite robust against such perturbations.

When the flux is a half of the flux quantum, the time-reversal symmetry is recovered but the channel number is even. In this case, the determinant of an antisymmetric matrix does not vanish and the perfect channel is destroyed. Therefore, the flux dependence in nanotubes is quite different from a $\phi_0/2$ oscillation in metallic systems on a cylinder surface, predicted theoretically\cite{31} and observed experimentally.\cite{32} This AAS oscillation arises due to the symmetry change caused
Fig. 5. Calculated inverse localization length as a function of flux $\phi$ in a metallic carbon nanotube. (a) Scatterers with a potential range larger than the lattice constant. (b) Effects of trigonal warping. (c) Scatterers with short-range potential. In (a) the inverse localization length vanishes identically for $\phi = 0$ due to the presence of a perfectly conducting channel. The localization effect appears for nonzero $\phi$ due to the symmetry crossover to the unitary, but takes a local minimum at the half flux quantum where the symplectic symmetry is recovered but states are localized because the channel number is even. In (b) the system has the unitary symmetry independent of the flux and the localization effect is essentially independent of the flux. In (c) the localization effect becomes weaker for nonzero flux except at the half flux quantum.

by the flux with period $\phi_0/2$. The symmetry crossover manifests itself in the flux dependence of the localization effect in metallic carbon nanotubes. Figure 5 compares the inverse localization length as a function of the flux when the system has the $S$ symmetry, when the $S$ symmetry is destroyed by a strong trigonal warping, and when the system has only the $T$ symmetry due to short-range scatterers.

4. Topological Anomaly and Berry’s Phase

An important feature is the presence of a topological singularity at $k = 0$. A neutrino has a helicity and its spin is quantized into the direction of its motion. The spin eigen
function changes its signature due to Berry’s phase under a \(2\pi\) rotation. Therefore the wave function acquires phase \(-\pi\) when the wave vector \(k\) is rotated around the origin along a closed contour.\(^{22,25,34}\) The signature change occurs only when the closed contour encircles the origin \(k=0\) but not when the contour does not contain \(k=0\). This topological singularity at \(k=0\) and associated Berry’s phase are the origin of the absence of backward scattering in metallic carbon nanotubes.\(^{22,24,25}\)

A singularity at \(\varepsilon = 0\) manifests itself in magnetic fields even in classical mechanics. The equation of motion gives the cyclotron frequency \(\omega_c = eBr^2/\varepsilon e\), where \(v\) is the electron velocity given by \(v = |v| = \gamma/h\). The cyclotron frequency \(\omega_c\) diverges and changes its signature at \(\varepsilon = 0\).\(^{7,14}\) In quantum mechanics \(\hat{k}_x\) and \(\hat{k}_y\) satisfy the commutation relation \([\hat{k}_x, \hat{k}_y] = -i/l^2\), where \(l\) is the magnetic length given by \(\sqrt{\hbar/eB}\). Semiclassically, the Landau levels can be obtained as \(\varepsilon_n = \pm \sqrt{n + 1/2} \hbar \omega_B\) with integer \(n\), where \(\hbar \omega_B = \sqrt{2}\gamma/l\). Because of the uncertainty relation, \(k^2 = 0\) is not allowed and there is no Landau level at \(\varepsilon = 0\). However, a full quantum mechanical treatment gives \(\varepsilon_n = \sqrt{|n|} \text{sgn}(n) \hbar \omega_B\), leading to the formation of Landau levels at \(\varepsilon = 0\).\(^{5}\)

We consider, for example, a system with scatterers with a potential range much smaller than the typical electron wavelength \((\text{which is actually infinite at } \varepsilon = 0)\).\(^{24}\) The relaxation time in the absence of a magnetic field becomes \(\tau_0^{-1} = 2\pi |\varepsilon| W/\hbar\) with \(W\) being a dimensionless parameter characterizing the scattering strength given by \(W = \langle n_1 u_0^2 \rangle/4\pi \gamma^2\), where \(u_i\) and \(n_i\) are the strength and the concentration of scatterers, respectively, and \(\langle \cdots \rangle\) means the average over impurities. With the use of the Boltzmann transport equation, the transport relaxation time becomes \(\tau(\varepsilon) = 2\tau_0(\varepsilon)\) and the conductivity \(\sigma_0 = (e^2/2\pi^2\hbar) W^{-1}\) independent of the Fermi level, i.e., nonzero even at \(\varepsilon = 0\) where the density of states vanishes.

In the presence of a magnetic field, the conductivity tensor \(\sigma_{\mu\nu}\) with \(\mu = x, y\) and \(\nu = x, y\) is given by \(\sigma_{xx} = \sigma_{yy} = \sigma_0/[1 + (\omega_c\tau)^2]\) and \(\sigma_{xy} = -\sigma_{yx} = -\sigma_0\omega_c\tau/[1 + (\omega_c\tau)^2]\). Using the explicit expressions for \(\omega_c\) and \(\tau\), we have \(\sigma_{xx} = \sigma_0 \xi_1/(1 + \xi_4)\) and \(\sigma_{xy} = -\sigma_0 \xi_2/(1 + \xi_4)\), with \(\xi_1 = 2\sqrt{\pi} W (\varepsilon_F/\hbar \omega_B)\). Because \(\tau(\varepsilon_F)^{-1} \propto |\varepsilon_F|\), the dependence on the Fermi energy \(\varepsilon_F\) is fully scaled by \(\hbar \omega_B\). Therefore, the conductivities exhibit a singular jump to zero at \(\varepsilon_F = 0\) from \(\sigma_0\) for nonzero \(\varepsilon_F\) in the limit of the vanishing magnetic field \(\hbar \omega_B \rightarrow 0\).

A singular behavior appears also in the dynamical conductivity.\(^{13}\) In a relaxation-time approximation, the dynamical conductivity is calculated as

\[
\sigma(\omega) = \frac{e^2}{8\hbar} \left[ \frac{4}{\pi} \frac{\varepsilon_F}{\hbar \omega + i|\hbar/\tau(\varepsilon_F)|} + 1 + i \frac{\ln h \omega + i|\hbar/\tau(\varepsilon_F)| - 2\varepsilon_F}{\pi} \right].
\]

(8)

Because \(\hbar/\tau(\varepsilon) \propto |\varepsilon|\), the frequency dependence is scaled by \(\hbar \omega/\varepsilon_F\). The scaling of the dynamical conductivity \(\sigma(\hbar \omega/\varepsilon_F)\) shows that \(\sigma(\omega, \varepsilon_F)\) exhibits a singular behavior at the point \((\omega, \varepsilon_F) = (0, 0)\). The correct way is to let \(\omega \rightarrow 0\) at each \(\varepsilon_F\), leading to a singular jump of the static conductivity to \(e^2/8\hbar\) at \(\varepsilon_F = 0\) from \(\sigma_0\) for nonzero \(\varepsilon_F\).
5. Quantum Transport

A more refined treatment has been performed for the density of states and the conductivity in a self-consistent Born approximation in which level-broadening effects are properly taken into account. Figure 6 shows an example. The result shows that the conductivity at $\varepsilon_F = 0$ is given by $e^2/\pi^2\hbar$, which is universal and independent of the scattering strength. The resulting conductivity varies smoothly across $\varepsilon_F = 0$ but exhibits a sharp jump in the limit of weak scattering ($W \ll 1$) from the Boltzmann result $\sigma_0$ for $\varepsilon \neq 0$ down to $\sigma = e^2/\pi^2\hbar$ at $\varepsilon_F = 0$. A similar calculation for a bilayer graphene shows much smoother variation with energy with minimum value $\sigma = 2(e^2/\pi^2\hbar)(1 + W^{-1})$ weakly dependent on the scattering strength.

The scattering probability $h/\tau(\varepsilon_F)$ is proportional to the final-state density of states with a coefficient independent of $\varepsilon_F$. Because the density of states is proportional to $\varepsilon_F$, the relaxation time is inversely proportional to $\varepsilon_F$. As a result the mobility is inversely proportional to $n_e \varepsilon_F^2$, leading to the conductivity independent of the Fermi energy and the electron concentration. Quite recently, charged-impurity scattering and screening effect were considered. In this case, the matrix element itself is proportional to the inverse of the Fermi energy both in the presence and absence of screening. Consequently, the low-temperature mobility becomes independent of the electron concentration and the conductivity increases in proportion to the electron or hole concentration $n_s$. 
Figure 7 shows some examples of the results. The frequency dependence is scaled by $\hbar \omega/\varepsilon_F$ as long as $\varepsilon_F \neq 0$. When $\varepsilon_F$ is very close to 0, however, the conductivity at $\omega = 0$ becomes small and the discrete jump present in the Boltzmann conductivity is removed. The energy scale causing this crossover behavior becomes smaller for weaker $W$ leading to a singular behavior of the dynamical conductivity in the weak scattering limit.

Figure 8 shows the density of states, the diagonal conductivity $\sigma_{xx}$, and the Hall conductivity $\sigma_{xy}$ calculated in the self-consistent Born approximation. In high magnetic fields, the broadening of the Landau level $n$ becomes $\hbar \omega_B \sqrt{2W(1 + \delta_{n0})}$ and the peak value of $\sigma_{xx}$ becomes $|(n| + \delta_{n0})(e^2/\pi^2 \hbar)$. It is interesting that the conductivity at $\varepsilon = 0$ remains $e^2/\pi^2 \hbar$ independent of the magnetic field. Further, the Hall conductivity in the gap between adjacent Landau levels is quantized into $4(j + 1/2)(e^2/\hbar)$ with integer $j$. Localization effects are not included in the self-consistent Born approximation, but it is clear that the Hall conductivity is quantized into these integers when states are localized in sufficiently high magnetic fields.

These quantized values are in agreement with the recent experimental observation. However, the observed minimum conductivity at zero energy seems to be 3~4 times as large as the predicted conductivity in the self-consistent Born approximation. It is difficult to discuss this conductivity in the vicinity of zero energy assuming realistic charged-impurity scattering, because a self-consistent determination of the screening and the density of states is necessary. Further, the linear screening may not be valid near $\varepsilon \approx 0$. The behavior near $\varepsilon \approx 0$ is expected...
to remain as an interesting and challenging subject.

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