Hall plateau diagram for the Hofstadter butterfly energy spectrum

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We extensively study the localization and the quantum Hall effect in the Hofstadter butterfly, which emerges in a two-dimensional electron system with a weak two-dimensional periodic potential. We numerically calculate the Hall conductivity and the localization length for finite systems with the disorder in general magnetic fields, and estimate the energies of the extended levels in an infinite system. We obtain the Hall plateau diagram on the whole region of the Hofstadter butterfly, and propose a theory for the evolution of the plateau structure in increasing disorder. There we show that a subband with the Hall conductivity $n e^2/h$ has $|n|$ separated bunches of extended levels, at least for an integer $n \leq 2$. We also find that the clusters of the subbands with identical Hall conductivity, which repeatedly appear in the Hofstadter butterfly, have a similar localization property.

I. INTRODUCTION

A two dimensional (2D) electron systems with a 2D periodic potential is expected to exhibit an intricate energy spectrum in a strong magnetic field, which is called the Hofstadter butterfly. When a magnetic length is the order of the lattice constant, the interplay of the Landau quantization and Bragg’s reflection yields a fractal-like series of the energy gaps, which depends sensitively on the number of magnetic flux quanta per a unit cell [1],

$$\phi = \frac{B a^2}{h/e},$$

where $B$ is the amplitude of the constant magnetic field $a$ is the lattice constant. Moreover it is shown that each subband carries a quantized Hall conductivity, which varies with the energy gaps in a non-trivial manner [2].

From the theoretical side, it is intriguing to consider the magnetotransport in this intricate energy spectrum. Broadening of the density of states [5, 6] and the conductivity [5] were investigated for the Hofstadter butterfly within the self-consistent Born approximation. For the localization regime, we expect that, by analogy with the unmodulated 2D systems, the extended levels appear only at certain energies in the Hofstadter spectrum, and $\sigma_{xy}(E)$ would turn into a series of the Hall plateaus separated by those energies.

The evolution of the extended states as a function of the disorder was proposed for several flux states in the Hofstadter butterfly [7–9]. A finite-size scaling analysis was performed for a 2D system modulated by a weak periodic potential and it was found that the modulation does not change the critical exponent at the center of the Landau level [10, 11].

II. FORMULATION

We first prepare the formulation to describe a Bloch electron in magnetic fields. Let us consider a two-dimensional system in a uniform magnetic field with a periodic potential $V_p$ and a disorder potential $V_d$.

$$H = \frac{1}{2m} (p + eA)^2 + V_p + V_d.$$  \hspace{1cm} (1)

We assume that $V_p$ has a square form

$$V_p = V \cos \frac{2\pi}{a} x + V \cos \frac{2\pi}{a} y.$$
and the disorder potential is composed of randomly distributed delta-potentials $\pm V_0$ with the number per unit area $n_z$, where the amounts of the positive and negative scatterers are taken to be equal. We consider only the lowest Landau level ($N = 0$), assuming that the magnetic field is strong enough and the mixing of the Landau levels is neglected. In the Landau gauge, the basis can be taken as

$$|0, k_y\rangle = \frac{1}{\sqrt{\pi l L_y}} \exp(ik_y y) \exp\left[-\frac{(x + k_y l)^2}{2l^2}\right], \quad (2)$$

with the magnetic length $l = \sqrt{\hbar/eB}$. The matrix elements of $V_y$ is then written

$$\langle 0, k_y' | V_y | 0, k_y \rangle = \delta_{k_y', k_y} 2Ve^{-\phi} \cos \frac{k_y a}{\phi} + (\delta_{k_y', k_y + 2\pi/a} + \delta_{k_y', k_y - 2\pi/a}) Ve^{-\phi}. \quad (3)$$

In an ideal system ($V_d = 0$), the wave function can be then expanded as

$$\psi_{k_y}(r) = \sum_m e_m(k_y)|0, k_y - \frac{2\pi}{a}|m\rangle, \quad (4)$$

with Bloch wave number $k_y$ ranging from $-\pi/a$ to $\pi/a$. The Schrödinger equation is then reduced to Harper's equation,

$$V e^{-\phi/2\pi} (c_{m+1} + c_{m-1}) + 2V e^{-\phi/2\pi} \cos \left(\frac{k_y a - 2\pi m}{\phi}\right) c_m = E c_m. \quad (5)$$

In a rational flux $\phi = p/q$ ($p, q$: co-prime integers), the equation becomes periodic in $m$ with a period $p$, which corresponds to a distance $qa$ in the center coordinate along $x$ axis, so that we have the Bloch condition $c_{m+p} = e^{ik_x qa}c_m$ with another Bloch wave number $k_x$ from $-\pi/(qa)$ to $\pi/(qa)$. As a result, we have $p$ independent states for each of $k = (k_x, k_y)$ in the Landau level, so that we can label the wave function as $\psi_{nk}$ on $q$-folded Brillouin zone with the subband index $n = 1, 2, ..., p$. We can decompose the wavefunction as $\psi_{nk}(r) = e^{ik_x x + k_y y} u_{nk}(r)$, where $u$ satisfies the magnetic Bloch condition

$$u(x, y + b) = u(x, y)$$
$$u(x + qa, y) = e^{2\pi i y/a} u(x, y). \quad (6)$$

Fig. 1 shows the energy spectrum in the lowest Landau level in $V_d = 0$ plotted against $\phi$. The intricate band structure is due to the number of subbands $p$, which is not a continuous function in $\phi$. We can show that $p$ subbands never overlap so that we always have $p - 1$ energy gaps inside the Landau level. The total width of the spectrum scales with the factor $e^{-\phi/2\pi}$ in eq. (5), and shrinks as the flux becomes smaller. The lower panel shows the zoom out of the spectrum covering from $\phi = 0$ to $10$. We can see that, as going to a higher field, a series of subbands splits away from the center toward the higher and lower energies. These are identified in the semi-classical picture as the quantization of the electron motion along the equi-potential line around the bottom or the top of the periodic potential. The widths of those levels become narrower for larger $\phi$ because the coupling between different potential valleys becomes exponentially small as the magnetic length becomes smaller.

The Hall conductivity $\sigma_{xy}$ is calculated by Kubo formula as,

$$\sigma_{xy} = \sum_{\epsilon_\alpha < E_F} \sum_{\epsilon_\beta \neq \epsilon_\alpha} \frac{\langle \alpha | v_x | \beta \rangle \langle \beta | v_y | \alpha \rangle - \langle \alpha | v_y | \beta \rangle \langle \beta | v_x | \alpha \rangle}{(\epsilon_\alpha - \epsilon_\beta)^2}, \quad (7)$$

where $E_F$ is the Fermi energy and $\epsilon_\alpha$ the energy of the eigenstate $|\alpha\rangle$ in the lowest Landau level. This is rewritten in an ideal system as [2]

$$\sigma_{xy} = \frac{e^2}{2\pi^2} \sum_n \int_{E < E_F} d^2k \Im \left[ \left( \frac{\partial u_{nk}}{\partial k_x} - \frac{\partial u_{nk}}{\partial k_y} \right)^2 \right], \quad (8)$$

where the summation is taken over all the occupied states. It is shown that the contribution from all the states in one subband is always quantized in units of $-e^2/h$ [2]. So we have an integer Hall conductivity when the Fermi energy $E_F$ is in every gap, since it simply becomes the summation of the integers for all the subbands below $E_F$. In such a gapped situation, we have a useful expression for the Hall conductivity called the Ströeda formula [13], which is directly derived from the Kubo formula (7). This is written as

$$\sigma_{xy} = -\epsilon \frac{\partial n}{\partial B}. \quad (9)$$

where $n$ is the number of states below the Fermi energy per unit area and $B$ is the magnetic field. In Fig. 1, we put the value of the Hall conductivity for each of gaps obtained by this formula. The contribution by one subband, $\Delta \sigma_{xy}$, is obtained as the difference in $\sigma_{xy}$ between the gaps above and below the subband. In the panel (b), the series of branches mentioned above always carries $\Delta \sigma_{xy} = 0$, since those subbands, coming from the localized orbitals around the potential minima or maxima, contain the constant number of states $n = 1/a^2$ which is independent of the magnetic field.

### III. LOCALIZATION AND THE QUANTUM HALL EFFECT

We now move on to the disordered system to investigate the localization and the quantum Hall effect. Here we numerically diagonalize the Hamiltonian (1) of finite systems of $L \times L$ with $L = Ma$ ($M$: integer), and calculate the localization length $L_{loc}$ and the Hall conductivity $\sigma_{xy}$. The localization length is obtained from the system.
The relation between $\Gamma$ and $\gamma$ is given by

$$\gamma = \frac{\pi}{2} \frac{\Gamma^2}{Wq^2}. \quad (11)$$

As a typical result, we show the calculation for the disordered system in the flux $\phi = 3/2$ in Fig. 2. The top panel indicates the system size dependence of $\sigma_{xy}$ with the density of states, and the bottom the inverse localization length $1/L_{loc}$. The parameter of the disorder is set to $\Gamma/V = 0.25$. In the clean limit, a Landau level splits into three separated subbands since $p = 3$, and the Hall conductivity carried by each of subbands is $\Delta \sigma_{xy} = 1, -1, 1$. Here and in the following we show $\sigma_{xy}$ and $\Delta \sigma_{xy}$ in units of $\epsilon^2/\hbar$. When the system is disordered we find that the gaps between the subbands are smeared out while the Hall conductivity converges to quantized values around the DOS dips, indicating the appearance of the Hall plateau. If we look at the size dependence of the Hall conductivity in the whole energy region, we see that $\sigma_{xy}$ always approaches 1 in the area $\sigma_{xy} > 1/2$ in increasing the sizes, and 0 in $\sigma_{xy} < 1/2$, so we expect that in an infinite system the continuous function $\sigma_{xy}(E)$ changes into the Hall plateaus connected by the steps, as shown as a step-like line. Therefore we speculate that the points of $\sigma_{xy} = 1/2$ are identified as the extended levels in an infinite system, and they indeed agree with the energies where localization length diverges as shown in (b).

We can see similar tendencies in the size dependence of $\sigma_{xy}$ in other fluxes as well, where the fixed points are always found at $\sigma_{xy} = n + 1/2$ ($n$: integer) in all the cases investigated. While each of three subbands has one extended level at a certain energy in $\phi = 3/2$, the localization generally depends on the Hall conductivity assigned to subbands. We have shown that a subband carrying zero Hall conductivity is all localized in finite disorder strength [12].
larger flux $\phi = 5/3$, as shown in Fig. 4. In clean limit, we have 5 subbands carrying $\Delta \sigma_{xy} = -1, +2, -1, +2, -1$ from the bottom to the top. Since the second and the fourth bands pass through $n + 1/2$ twice, each of them comes to have two extended levels in sufficiently small disorder on the present assumption. As going to the stronger disorder, one of two extend levels experiences a pair annihilation with the lowest or highest subband, and only three extended levels are left as in $\phi = 3/2$. When we start with a more complicated flux, we speculate that the Hall plateau structure is simplified one by one, going through analogs of the simpler fluxes near around.

To check that there are actually two separated extended levels in a subband carrying $\Delta \sigma_{xy} = 2$, we can make a scaling analysis in the Hall conductivity with varying the system size. In the flux $\phi = 5/3$, unfortunately, the large statistical error prevents us from resolving the splitting, but we can in a similar situation in other flux $\phi = 8/15$, where the lowest subband carrying $\Delta \sigma_{xy} = 2$. Fig. 5 shows the Hall conductivity and the localization length for the lowest subband with disorder $\Gamma = 0.0173V$. The difference in $\sigma_{xy}$ between $L/a = 45$ and 30, shown in the middle panel, indicates that $\sigma_{xy}$ actually has turning points around $\sigma_{xy} = 1/2$ and 3/2. This is confirmed by the calculation of the localization length in the lowest panel, showing that $L_{loc}$ diverges at those two energies, so it is quite likely that there are two extended levels in this subband in an infinite system.

A major difference from $\phi = 5/3$ is that the ideal subband width is much narrower than the energy gap in the present case, so we can set the disorder in such a way that the states in the subband are completely mixed up but not with other subbands. We speculate that those situation makes the localization length smaller and enables us to resolve the separated extended levels in a finite size calculation. We have another example where one subband has more than one extended levels in anisotropic modulated quantum Hall systems. There $\sigma_{xy}(E)$ in a subband with $\Delta \sigma_{xy} = 1$ has a non-monotonic behavior crossing $\sigma_{xy} = 1/2$ three times, so that three extended levels arise [12].

Lastly we investigate a flux $\phi = 8/7$, slightly away from $\phi = 1$. Around this region the Landau level splits into a number of tiny subbands as seen in the original Hofstadter butterfly, just like Landau levels in usual 2DEG around the zero field. This is actually understood as the Landau quantization in the magnetic Bloch band at $\phi = 1$ caused by the residual flux $\phi = 1$ [16]. As seen in Fig. 6, each of those 'Landau levels' equally carries the Hall conductivity by $\Delta \sigma_{xy} = 1$ while the center one has a large negative value. It is highly nontrivial how the extended states evolves in increasing disorder in such a situation.

Fig. 6 shows the disorder dependence of density of states, the Hall conductivity $\sigma_{xy}$ and the traces of $\sigma_{xy} = n + 1/2$, in a fixed system size $L = 14a$ in $\phi = 8/7$. Remarkably the result suggests that a pair creation of the extended states can occur in increasing disorder, as

IV. EVOLUTION OF THE EXTENDED LEVELS

When we consider a situation where the disorder increases in the Hofstadter butterfly, we expect that the extended levels in the subbands move and merge on the energy axis in some way, and in large enough disorder, only one remains at the center of Landau level. It would be nontrivial and intriguing to ask how they evolve as a function of disorder, and how different they are in various fluxes. Here we study the evolutions for some particular fluxes, on the assumption that the extended levels always exist at the energies of $\sigma_{xy} = n + 1/2$ with integers $n$, and the region where $n - 1/2 < \sigma_{xy} < n + 1/2$ in finite systems becomes $\sigma_{xy} = n$ in an infinite system.

We first show the results for the flux $\phi = 3/2$ in Fig. 3. The upper and middle panel indicate the density of states and the Hall conductivity, respectively, in a finite system $L = 12a$ for several disorder parameters $\Gamma$. The lower shows the traces of the extended levels obtained by just taking the energies of $\sigma_{xy} = n + 1/2$. We observe that three extended levels get closer as the disorder increases, and contract into one at a certain $\Gamma$. The combination of three branches at a time is due to the electron-hole symmetry with respect to $E = 0$, owing to the equal amount of the positive and negative scatterers. If we break the symmetry by introducing the imbalance between them, the evolution changes in such a way that two of them annihilates and one is left intact [15].

The situation becomes a little complicated in a slightly smaller flux $\phi = 5/3$, as shown in Fig. 4. In clean limit, we have 5 subbands carrying $\Delta \sigma_{xy} = -1, +2, -1, +2, -1$ from the bottom to the top. Since the second and the fourth bands pass through $n + 1/2$ twice, each of them comes to have two extended levels in sufficiently small disorder on the present assumption. As going to the stronger disorder, one of two pair levels experiences a pair annihilation with the lowest or highest subband, and only three extended levels are left as in $\phi = 3/2$. When we start with a more complicated flux, we speculate that the Hall plateau structure is simplified one by one, going through analogs of the simpler fluxes near around.

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Our theory may suggest a somewhat different story where pair creations of the extended states occur, since we actually observe similar dips in $\sigma_{xy}(E)$ around higher Landau levels in disordered non-periodic 2DEG.

We also remark that the evolution of the extended levels generally depends on the correlation lengths of the disorder potential. It was shown in the tight-binding lattice models, on the other hand, that the the states with the negative Hall conductivity move down from the tight-binding band center, to annihilate with the extended states in the lower Landau levels [20–22].

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FIG. 5: (a) Hall conductivity $\sigma_{xy}$ and the density of states $\rho$ calculated for the lowest subband in $\phi = 8/15$, carrying $\Delta \sigma_{xy} = -2e^2/h$, with the disorder $\Gamma/V = 0.0173$ and various sizes $L/a = 30, 45$, and 60. The gray step-like line is an estimate in $L \to \infty$. The vertical line at $E \approx -0.0865$ represents the energy region of the ideal subband. (b) Relative value of $\sigma_{xy}$ in $L/a = 60$ estimated from $L/a = 45$. (c) Inverse localization length (in units of $a$) estimated from the Thouless number. Two vertical lines penetrating the panels represent the energies of the extended levels in an infinite system.

FIG. 6: Plots similar to Fig. 3 calculated for $\phi = 8/7$. Arrows indicate the positions of the pair creations of the extended levels.

FIG. 7: Schematic figure of a pair creation of the extended levels. The left and right panels show $\sigma_{xy}(E)$ in the smaller and larger disorder, respectively, where a solid curve is for a finite system and a gray line for an infinite system. The pair of the extended levels are newly created when a dip touches the line of $\sigma_{xy} = 0.5(-e^2/h)$.

FIG. 8: The Landau level corresponds to two major plateaus with the energies of $\sigma_{xy} = n + 1/2$ in a finite system. In the density of states (b), we recognize valleys in the contour plot as the remnants of the minigaps, and some of them can be associated with the Hall plateaus in (a). For instance, a pair of valleys between $\phi = 1$ and 2 inside the Landau level corresponds to two major plateaus with $\sigma_{xy} = 0$ in $0 < E < 0.5V$ and $\sigma_{xy} = 1$ in $-0.5V < E < 0$. The localization length is smaller for larger $\phi$ in those branches, mainly because the mixing of the states becomes stronger in narrower subbands.

In Fig. 10, we show the Hall plateau diagrams and the density of states for the different disorder parameters $\Gamma/V = 0.15$ and 0.5, which exhibit the dependence on $\Gamma$ together with Fig. 9 (a) and (b). We can see that the small Hall plateaus coming from the fine gap structure gradually disappear as the disorder becomes larger, and
the only extended level is left at the center of the Landau level in the strong disorder limit. We here notice in $\Gamma/V = 1.5$ that two largest plateaus between $\phi = 1$ and 2 mentioned above, are more easily destroyed around $\phi = 1$ than around $\phi = 2$, or the right end is detached from $\phi = 1$ while the left sticks to $\phi = 2$. This is because the tiny gaps around $\phi = 1$ is easily swallowed up by the large density of states around the center.

VI. SELF-SIMILARITY

The interesting observation in the Hofstadter butterfly is that the identical spectrum with the identical distribution of the Hall conductivity repeatedly appears in different fluxes. In Fig. 1 (b), we can see that the spectrum and the Hall conductivity at $\phi$ corresponds to the middle part of $\phi + 2$ with the top and bottom branches excluded, so that the entire structure repeats over $\phi + 2m$ with integers $m$. The proof for the correspondence is presented in Appendix. We note that, while the Hofstadter butterfly has similar gap structures everywhere in a fractal fashion, the Hall conductivities in the corresponding gaps do not always coincide. An example of the incomplete correspondence is seen between $\phi$ and $\phi' = 1/(1-\phi)$ (such as $\phi = 3$ and $3/2$). In the tight binding model, it was shown that the distribution of the Hall conductivity within a cluster are resembled up to a scale factor among some series of fluxes [27].

In the following we show that, in the presence of disorder, the corresponding clusters with identical Hall conductivity have qualitative agreement also in the localization length. Here we particularly take a pair of fluxes $\phi = 3/2$ and $7/2$, in both of which the central three subbands have the Hall conductivity (1,1,1). In Fig. 11 we compare the disorder effects on the density of states and the localization length. We set the disorder as $\Gamma = 0.25$ and $0.20$ for $\phi = 3/2$ and $7/2$, respectively, so that the renormalized DOS broadening $\gamma/W_{tot}$ is equivalent, where we defined $\gamma$ by putting in eq. (11) the full width of three subbands $W_{tot}$. The result shows that the densities of states are broadened equivalently as expected, and that the localization length $L_{loc}$ then agree qualitatively without any scale factors.

Fig. 11 (c) shows the evolution of the extended levels as the function of the disorder, which are obtained by taking the points of $\sigma_{xy} = 1/2$. Three extended levels in $\phi = 3/2$ and $7/2$ evolve in a parallel fashion with the disorder strength $\gamma/W_{tot}$, where they come closer as $\gamma$ becomes larger and combine into one at $\gamma/W_{tot} \approx 0.4$. The critical disorder at which the combination occurs is slightly smaller in $\phi = 7/2$ than in $3/2$, presumably be-
cause in $\phi = 7/2$ the level repulsion from the outer subbands (out of the figure) pushes the states in the central three subbands towards the center of the spectrum and that enhances the contraction of the extended levels. We expect that the critical behavior of the three extended levels at the combing point is universal, but we could not estimate the critical exponent in this simulation due to statistical errors. The evolution of the plateau diagram should become basically similar among $\phi + 2m$, so we know all from the information of the first unit.

The similar subband structures with the identical Hall conductivity can be found in other hierarchies in the Hofstadter butterfly. The smallest unit that we have found is schematically shown in Fig. 12, where the gap structure and the Hall conductivity inside each gap coincide. The letter ‘A’ indicates the direction.

weak enough that the mixing among different units can be neglected.
VII. CONCLUSION

We studied the quantum Hall effect in a Landau level in presence of a two-dimensional periodic potential with short-range disorder potentials. It is found that, in all the cases we studied, the Hall conductivity becomes size independent at $\sigma_{xy} = n + 1/2$ (in units of $-e^2/h$), and those points are identified as the extended levels in an infinite system. We propose a possible model for the evolution of the extended levels by tracing $\sigma_{xy} = n + 1/2$, which predicts that a subband with $\sigma_{xy} = n$ has $n$ (or more) bunches of the extended levels, and possibly that a pair creation of the extended level can occur in increasing disorder, as well as a pair annihilation. We also find that the clusters of the subbands with an identical Hall conductivity, which compose the Hofstadter butterfly in a fractal fashion, have a similar localization length in the presence of the disorder.

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APPENDIX: CONCINDEENCE IN THE HALL CONDUCTIVITY

The equivalence in the gap structure with the Hall conductivity between $\phi$ and $\phi + 2$, discussed in Sec. VI, is explained as follows. The number of subbands in a Landau level is given by the numerator of the magnetic flux $\phi$, so that we have $p$ bands in $\phi = p/q$ and $p + 2q$ bands in $\phi + 2$. Each single subband in $\phi$ and $\phi + 2$ consists of the equal number of states per unit area, $1/(qa^2)$, since two fluxes have a common denominator $q$ so that they have equal foldings of the Brillouin zone. We can then see that each of the top and bottom branches in $\phi + 2$ contains $q$ subbands, because each has a constant number of states $1/a^2$ as explained in Sec. II. Thus the number of subbands in the middle part in $\phi + 2$ (the top and bottom branches removed) becomes $(p + 2q) - 2q = p$, which is equal to the total subbands in $\phi$. Now we see that the corresponding spectra between $\phi$ and $\phi + 2$ have the same number of subbands with the equal number of states, so we come to the conclusion that the Hall conductivity becomes identical between the corresponding gaps, by using the Strˇ eda formula (9).