Transport in Bilayer Graphene

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The transport properties of a bilayer graphene are studied theoretically within a self-consistent Born approximation. The electronic spectrum is composed of $k$-linear dispersion in the low energy region and $k$-square as in an ordinary two-dimensional metal in the high energy, leading to a crossover between different behaviors in the conductivity with changing the Fermi energy or disorder strengths. We find that the conductivity approaches $2e^2/\pi^2\hbar$ per spin in the strong disorder regime, independently of the short- or long-range disorder.

I. INTRODUCTION

Recently there was an experimental development in fabrication of atomically thin graphene, or a single layer graphite, that enables us to access its exotic electronic properties.\textsuperscript{1–3} The magnetotransport was measured and the integer quantum Hall effect was observed.\textsuperscript{4–6} In the experiments, a multilayer which contains a few graphene sheets are also available.\textsuperscript{4,6} The electronic structure of the bilayer graphene was studied theoretically and the spectrum was found to be essentially different from that of a monolayer.\textsuperscript{7} The purpose of this paper is to study transport properties of the bilayer graphene.

A graphite monolayer has a $k$-linear, massless Dirac-like spectrum and has long attracted theoretical interests as a ‘relativistic’ problem in the condensed matter physics, where $k$ is the absolute value of the wave vector. The theoretical studies for the transport in such an exotic electronic structure have been given by several authors, where the conductivity with/without magnetic fields,\textsuperscript{8} the Hall effect,\textsuperscript{9–11} quantum corrections to the conductivity,\textsuperscript{12} the dynamical transport\textsuperscript{13} are investigated. The results show that the conductivity exhibits various singular behaviors in the vicinity of zero energy.\textsuperscript{8,9,13–15}

In the bilayer graphene the energy dispersion includes both $k$-linear and $k$-square terms. In this paper, we calculate the diagonal conductivity in the absence of a magnetic field within a self-consistent Born approximation (SCBA). A similar SCBA analysis has been applied for the monolayer graphene.\textsuperscript{8,9,13} We shall find that the coexistence of $k$-linear and $k$-square dispersions leads to a crossover between the transport properties like a monolayer graphene and like an ordinary two-dimensional metal as the Fermi energy is changed. The conductivity becomes nearly universal, $2e^2/\pi^2\hbar$ per spin, in the case of a large disorder. The analysis is made for two different kind of scatterers, short-range and long-range, where the former represents on-site random energies distributed on the carbon atoms and the latter the slowly-varying random potential compared to the lattice constant but shorter than the typical electron wavelength.

In Sec. II, an effective Hamiltonian and resulting energy spectrum in a bilayer graphene are discussed, and model scatterers are introduced. The self-consistent Born approximation is briefly described in Sec. III. Explicit results are discussed for short-range scatterers in Sec. IV and for long-range scatterers in Sec. V. A discussion and a brief summary are given in Sec. VI.

II. EFFECTIVE HAMILTONIAN AND ENERGY SPECTRUM

The bilayer graphene is composed of a pair of hexagonal networks of carbon atoms, which include $A$ and $B$ atoms on the top layer and $A'$ and $B'$ on the bottom, as schematically shown in Fig. 1. Two layers are arranged in the AB stacking, where $A$ atoms are located above $B'$ atoms, and $B$ or $A'$ atoms are above or below the center of hexagons in the other layers. The unit cell contains four atoms $A, B, A', B'$, and the Brillouin zone becomes identical with that of the monolayer graphene. We model the system by a tight-binding Hamiltonian based on the Slonczewski-Weiss-McClure graphite model.\textsuperscript{17,18} We include three parameters $\gamma_0$, $\gamma_1$, and $\gamma_3$, where $\gamma_0$ represents the intralayer coupling $A \leftrightarrow B$ or $A' \leftrightarrow B'$, and $\gamma_1$ and $\gamma_3$ the interlayer coupling $A \leftrightarrow B'$ and $B \leftrightarrow A'$, respectively. The coupling parameters are estimated as $\gamma_0 \approx 3.16$ eV,\textsuperscript{19} $\gamma_1 \approx 0.39$ eV,\textsuperscript{20} and $\gamma_3 \approx 0.315$ eV.\textsuperscript{21}

We can show that the low energy spectrum is given by
the states around $K$ and $K'$ points in the Brillouin zone. Neighboring $A$ and $B'$ sites are coupled by $\gamma_1$ to create the bonding and anti-bonding states away from the Fermi level, and the low energy states are given by remaining $A'$ and $B$ sites.\(^7\) The effective Hamiltonian reads,

$$\mathcal{H}_K = \frac{\hbar^2}{2m^*} \begin{pmatrix} 0 & k_x' \\ k_y' & 0 \end{pmatrix} - \frac{\hbar^2 k_0}{2m^*} \begin{pmatrix} 0 & k_+ \\ k_- & 0 \end{pmatrix},$$

(1)

$$\mathcal{H}_{K'} = \frac{\hbar^2}{2m^*} \begin{pmatrix} 0 & k_x' \\ k_y' & 0 \end{pmatrix} + \frac{\hbar^2 k_0}{2m^*} \begin{pmatrix} 0 & k_- \\ k_+ & 0 \end{pmatrix},$$

(2)

where $k_\pm = k_x \pm ik_y$ with $k$ being the wavevector measured from $K$ or $K'$ points, and the effective mass $m^*$ and the wavenumber $k_0$ are defined by

$$\frac{\hbar^2}{2m^*} = \frac{(\sqrt{3}a\gamma_0/2)^2}{\gamma_1},$$

(3)

and

$$k_0 = \frac{2}{\sqrt{3}a} \frac{\gamma_3\gamma_1}{\gamma_0},$$

(4)

with the lattice constant $a = 0.246$ nm. The $k$-linear term in the Hamiltonian (2) describes the direct hopping between $A'$ and $B$ sites, and the $k$-square term the second order process between $A'$ and $B$ via $A$-$B'$ dimers. A typical energy where the $k$-square and $k$-linear terms become comparable can be defined by

$$\varepsilon_0 = \frac{\hbar^2 k_0^2}{2m^*} = \left( \frac{\gamma_3}{\gamma_0} \right)^2 \gamma_1.$$  

(5)

The eigen energy of (2) becomes

$$\varepsilon_{jk} = \frac{\hbar^2}{2m^*} s k \sqrt{k^2 + 2k_0 k \cos 3\varphi + k_0^2},$$

(6)

where the upper sign corresponds to $j = K$ and the lower to $K'$, $s = \pm 1$, $k = \sqrt{k^2 + k_0^2}$, and $\varphi = \arg(k_0)$ with $\arg(z)$ being the argument $\varphi$ in $z = |z|e^{i\varphi}$. The eigenvectors corresponding to (6) are

$$\phi_{jk} = \begin{pmatrix} \phi_{jk}' \\ \phi_{jk}'' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\theta_{jk}} \\ s \end{pmatrix},$$

(7)

with

$$\theta_{Kk} = \arg(-k_0e^{i\varphi} + ke^{-2i\varphi}),$$

(8)

$$\theta_{K'k} = \arg(k_0e^{-i\varphi} + ke^{2i\varphi}).$$

(9)

Fig. 2 shows the energy dispersion (6) for $j = K$. In the high energy region $|\varepsilon| > (1/4)\varepsilon_0$, we have a single trigonally warped Fermi line, which becomes closer to a circle as going to the higher energy as the $k$-square term dominates in the Hamiltonian. The equi-energy line becomes convex for $\varepsilon \gtrsim 10.8\varepsilon_0$. In the low energy region $|\varepsilon| < (1/4)\varepsilon_0$, the Fermi line splits into four separated pockets for each of $K$ and $K'$, one center part and three satellite parts that are located trigonally. In the vicinity of zero energy $|\varepsilon| \ll \varepsilon_0$, the dispersion becomes linear in $k$-space with respect to four Fermi points, where the center pockets can be approximated as a circle with radius $k = \varepsilon/\varepsilon_0|k_0$ and three satellites as ellipses with the longer and shorter radii $|\varepsilon|/\varepsilon_0|k_0$ and $|\varepsilon|/\varepsilon_0|k_0/3$. While splitting of four Fermi lines occurs only in very low energies, the trigonal warping extends to much higher energy as seen in Fig. 2.

The velocity operator $v_x = (\partial H/\partial k_x)/h$ has nonzero matrix elements between the states on the identical $k$-points $(j,k)$ written as

$$(v_{jk})_{ss'} = \langle jks'|v_x|jks \rangle = sw_{jk} + s^*w_{jk}^*,$$

(10)

with

$$w_{Kk} = \frac{\hbar}{4m^*} (k_0 + 2ke^{i\varphi})e^{-i\theta_{Kk}},$$

(11)

$$w_{K'k} = \frac{\hbar}{4m^*} (-k_0 + 2ke^{i\varphi})e^{-i\theta_{K'k}}.$$  

(12)

In the high energy region $|\varepsilon| > (1/4)\varepsilon_0$, the mean square...
velocity averaged on the contour $\varepsilon = |\varepsilon_{jk}\rangle$ is given by
\[
\langle \langle |v_{jk}\rangle |^2 |\varepsilon = \frac{|\varepsilon|}{m^*} + \frac{\varepsilon_0}{2m^*} (1 + ss^*),
\]
where the first and second terms come from the $k$-square and $k$-linear terms in the dispersion, respectively. In the vicinity of zero energy $|\varepsilon| \ll \varepsilon_0$, we have
\[
\langle \langle |v_{jk}\rangle |^2 |\varepsilon = \frac{3\varepsilon_0}{4m^*}. \tag{14}\]

The density of states per spin is defined by
\[
\rho_0(\varepsilon) = \frac{1}{\Omega} \sum_{jks} \delta(\varepsilon - \varepsilon_{jk}),\tag{15}\]
where $\Omega$ is the area of the system. In the high energy region $|\varepsilon| > (1/4)\varepsilon_0$, this becomes
\[
\rho_0(\varepsilon) \approx \frac{m^*}{\pi h^2} \equiv \rho_\infty, \tag{16}\]
with the order of $O(\varepsilon_0^2/k^2)$ neglected. In the vicinity of zero energy $|\varepsilon| \ll \varepsilon_0$, on the other hand,
\[
\rho_0(\varepsilon) \approx \frac{4m^* |\varepsilon|}{\pi k^2 \varepsilon_0}. \tag{17}\]

The density of states diverges logarithmically at $|\varepsilon|/\varepsilon_0 = 1/4$ due to the presence of saddle points in the dispersion.

For the parameters mentioned above, $m^*/m_0 = 0.033$ with $m_0$ being the free electron mass, $k_0(2\pi/a)^{-1} = 2.2 \times 10^{-3}$, and $\varepsilon_0 \approx 3.9$ meV. The electron concentration corresponding to $(1/4)\varepsilon_0$ is $n_s = 1.7 \times 10^{10}$ cm$^{-2}$, that to $\varepsilon_0$ is $1.0 \times 10^{11}$ cm$^{-2}$, and that to $10.8\varepsilon_0$ is $1.1 \times 10^{12}$ cm$^{-2}$.

As the typical electron concentration in the present system is $10^{12}$ cm$^{-2}$ ($|\varepsilon|/\varepsilon_0 \sim 10$) or larger, the trigonal warping is appreciable, but it is extremely hard to realize the situation where four Fermi lines are well split from each other.

The Hamiltonian (2) is formally equivalent to that of the monolayer system, where the $k$-square term comes from a higher order term in the $\mathbf{k} \cdot \mathbf{p}$ approximation. However, the parameter $\varepsilon_0$ for the monolayer becomes the order of $\gamma_0$ and thus larger than in the bilayer by the order of 1000, and the $k$-square term gives only a small perturbation. In the bilayer, on the contrary, $k$-square becomes dominant in the dispersion at small energy $\varepsilon_0$. Note also that the Hamiltonian (2) becomes invalid when the energy becomes as high as the anti-bonding states of $A$-$B'$ dimers.

The deviation from (2) seems to become appreciable around $|\varepsilon| \sim \gamma_1/4$ ($\sim 0.1$ eV corresponding to $n_s \sim 2.7 \times 10^{12}$ cm$^{-2}$). The dispersion of the bilayer graphene is much closer to that of three-dimensional graphite, where we see a similar trigonal structure in the $k_xk_y$ plane with a fixed wavenumber $k_z$ along the stacking direction.

In terms of the eigen vector $\phi_{jk}\rangle$ the amplitude of the atomic orbital at sites $R_A'$ and that at $R_B$ are given by
\[
\psi_A(R_A') = \frac{1}{\sqrt{N}} \phi_{jk\rangle}^A \exp[i(K_j + k) \cdot R_A'], \tag{18}\]
\[
\psi_B(R_B) = \frac{\omega j}{\sqrt{N}} \phi_{jk\rangle}^B \exp[i(K_j + k) \cdot R_B], \tag{19}\]
for $j = K$ and $K'$, where $\omega K = \omega^{-1}$, $\omega K' = \omega$ with $\omega = \exp(2\pi i/3)$, and $N$ is the number of unit cells in the system.

Dominant scatterers in the present system have not been known well. In the following, we shall consider two kinds of model scatterers, short-range scatterers localized only on $B$ sites or on $A'$ sites and long-range scatterers, the potential of which spreads over a certain length scale larger than the lattice constant $a$. In each case we assume that the disorder strength is weaker than the dimer coupling $\gamma_1$, so that we need to consider only the potential on $A'$ and $B$ sites. The effective Hamiltonian for the disorder potential can be derived similarly as in the monolayer graphene, if we identify $A'$ and $B$ sites in the bilayer with $A$ and $B$ in the monolayer.\textsuperscript{24,25}

The matrix elements for the short-ranged potential are written as
\[
\langle j'k's'|U_i^A'|jk\rangle = \frac{u_i^A}{\Omega} \exp[\iota(k_j + k) \cdot R_A'] \phi_{jk\rangle}^A \phi_{j'k's'}^{A'} , \tag{20}\]
\[
\langle j'k's'|U_i^B'|jk\rangle = \frac{u_i^B}{\Omega} \exp[\iota(k_j + k) \cdot R_B] \phi_{j'k's'}^{B'} \phi_{jk\rangle}^B , \tag{21}\]
for scatterers at $R_A'$ and $R_B$, respectively, where $u_i^A$ and $u_i^B$ are the integrated intensity of the potential, i.e., the coefficient of the delta potential. For the long range disorder, we assume that the potential range is much larger than the lattice constant $a$ but smaller than the typical wavelength $2\pi/k$ with $k$ being the wavenumber from $K$ or $K'$ points. Then we can neglect the matrix elements for inter-valley scatterings between $K$ and $K'$, while those for intra-valley can be written as
\[
\langle j'k's'|U_i|jk\rangle = \delta_{j'j} \frac{u_i}{\Omega} \exp[\iota(k_j - k') \cdot R_i] \phi_{j'k's'}^B \phi_{jk\rangle}^B , \tag{22}\]
where $u_i$ is the integrated intensity of the potential.

We have another possibility for long-range disorder potential, where each of scatterers is effective only in one layer and smooth in the other layer. This is modeled by the long-range potential which has amplitudes only over either $A'$ or $B$ sites. The situation then becomes almost equivalent to the short-range case, while the only difference is that $K$ and $K'$ are decoupled, reducing the self-energy by the factor two.
III. SELF-CONSISTENT BORN APPROXIMATION

In the self-consistent Born approximation, the self-energy of the disorder-averaged Green’s function \( G_{\alpha,\alpha'} \) is given by\(^8\)

\[
\Sigma_{\alpha,\alpha'}(\varepsilon) = \sum_{\alpha_1,\alpha_1'} \langle U_{\alpha,\alpha_1} U_{\alpha_1',\alpha'} \rangle \langle G_{\alpha_1,\alpha_1'}(\varepsilon) \rangle,
\]

(23)

with \( \alpha = (j,k) \), where \( \langle \rangle \) represents the average over the imurity configurations. In the present system the conductivity becomes isotropic in spite of the presence of strong trigonal warping. It can be calculated by the Kubo formula,

\[
\sigma(\varepsilon) = \frac{\hbar e^2}{2\pi \Omega} \text{Re} \text{Tr} \left[ \langle v_x G^R v_x G^A \rangle - \langle v_x G^R v_x G^R \rangle \right],
\]

(24)

where \( G^R = (\varepsilon - \mathcal{H} + i0)^{-1} \) and \( G^A = (\varepsilon - \mathcal{H} - i0)^{-1} \) are the retarded and the advanced Green’s functions, respectively, with \( \mathcal{H} \) being the Hamiltonian including the disorder potential. This can be rewritten as

\[
\sigma(\varepsilon) = \frac{\hbar e^2}{2\pi \Omega} \text{Re} \text{Tr} \left[ v_x (G^R v_x G^A) - v_x (G^R v_x G^R) \right],
\]

(25)

with \( \tilde{v}_x R = \tilde{v}_x (\varepsilon + i0, \varepsilon - i0) \) and \( \tilde{v}_x A = \tilde{v}_x (\varepsilon + i0, \varepsilon + i0) \) satisfying

\[
\tilde{v}_x(\varepsilon, \varepsilon') = v_x + (UG(\varepsilon)\tilde{v}_x G(\varepsilon')U).
\]

(26)

In SCBA, \( \tilde{v}_x \) should be calculated in the ladder approximation. In the above and also hereafter we omit the summation over the spin degeneracy, so the actual conductivity should be multiplied by the factor two.

In the case of present model scatterers, the self-energy and therefore the averaged Green’s function become diagonal with respect to the wavenumber and the band index. Further, the self-energy is independent of the wavenumber and the band index and is determined by the energy alone. Thus we have

\[
\langle G_{\alpha,\alpha'}(\varepsilon) \rangle = \delta_{\alpha,\alpha'} G_{\alpha}(\varepsilon),
\]

(27)

\[
G_{\alpha}(\varepsilon) = G(\varepsilon, \varepsilon_{\alpha}) \equiv \frac{1}{\varepsilon - \Sigma(\varepsilon) - \varepsilon_{\alpha}},
\]

(28)

where \( \Sigma(\varepsilon) \) is the self-energy.

IV. SHORT RANGE SCATTERERS

For short-range scatterers, we assume that they are equally distributed to A and B sites with density \( n_A^U = n_A^B = n_A/2 \) and the identical mean square amplitude \( \langle |u_A^U|^2 \rangle = \langle |u_A^B|^2 \rangle = u_A^2 \). Then, the self-energy is given by

\[
\Sigma(\varepsilon) = \frac{n_A u_A^2}{4\Omega} \sum_{\alpha} G_{\alpha}(\varepsilon).
\]

(29)

By substituting the summation over \( j \) and \( \mathbf{k} \) with the integration in energy \( \varepsilon' = |\varepsilon,j,k| \), we can rewrite this into

\[
\Sigma(\varepsilon) = \frac{n_A u_A^2}{2} \int_0^\infty d\varepsilon' \rho(\varepsilon') \sum_{\alpha} G(\varepsilon, \varepsilon').
\]

(30)

The density of states is given by

\[
\rho(\varepsilon) = -\frac{1}{4\pi n_A u_A^2} \int_0^\infty d\varepsilon' \rho(\varepsilon + i0) = -\frac{4}{\pi n_A u_A^2} \text{Im} \Sigma(\varepsilon + i0).
\]

(31)

For the conductivity, we can show that the vertex correction vanishes in the short range scatterers, or \( \tilde{v}_x = v_x \), and obtain

\[
\sigma(\varepsilon) = \frac{\hbar e^2}{2\pi \Omega} \text{Re} \sum_{jks's'} \langle (v_{jk})_s v_{s's'}^A \rangle G_{jk}(\varepsilon + i0) G_{jks'}(\varepsilon + i0) \rangle - \text{Re} \sum_{jks's'} \langle (v_{jk})_s v_{s's'}^A \rangle G_{jk}(\varepsilon - i0) G_{jks'}(\varepsilon - i0) \rangle.
\]

(32)

In the second line we have substituted the summation over \( j \) and \( \mathbf{k} \) with the integral in \( \varepsilon' = |\varepsilon,j,k| \).

We first look on the Boltzmann limit by taking \( \Sigma \to 0 \), which should be valid in a weak disorder case satisfying \( |\varepsilon| \gg |\Sigma| \). By assuming that \( \Sigma \) is infinitesimal in (30), we can calculate \( \Sigma \) explicitly as

\[
\Sigma(\varepsilon + i0) = -i\frac{\pi}{2} \varepsilon_0 W \frac{\rho_0(\varepsilon)}{\rho_\infty} \equiv -i\frac{\hbar}{2\tau},
\]

(33)

where \( \tau \) is the lifetime and the dimensionless parameter \( W \) is defined by

\[
W = \frac{n_A u_A^2 \rho_\infty}{\varepsilon_0}.
\]

(34)

For sufficiently high energy, \( \hbar/2\tau \approx (\pi/2)\varepsilon_0 W \) showing that \( W \sim 1 \) characterizes a typical disorder strength which smears out the fine low energy structure due to the \( k \)-linear term. This \( W \) is the same as that defined in Ref. 13 and as \( A^{-1} \) defined in Ref. 8.

For the conductivity, we take only the terms with \( G^R G^A \) and \( s = s' \) as a dominant contribution in (32), and obtain a familiar form,

\[
\sigma(\varepsilon) \approx e^2 \rho_0 \tau (v_x^2)_{\varepsilon}.
\]

(35)

Here \( (v_x^2)_{\varepsilon} = \langle |(v_{jk})_s|^2 \rangle_\varepsilon \) is the Fermi-surface average of the diagonal matrix element of \( v_x \) and use has been made of the approximation

\[
\frac{1}{\Omega} \sum_{\alpha} G_{\alpha}(\varepsilon) G_{\alpha}(\varepsilon) \approx \frac{2\pi \tau}{\hbar} \rho_0.
\]

(36)

With the use of (13) the conductivity for the higher energy \( |\varepsilon| > \varepsilon_0 \) becomes

\[
\sigma(\varepsilon) = \frac{e^2}{\pi \hbar W} \left( \frac{|\varepsilon|}{\varepsilon_0} + 1 \right).
\]

(37)
The term proportional to $|\varepsilon|$ results from the $k$-square term of the dispersion and is rewritten as $e^2 \rho_\infty |\varepsilon|/\tau^* m^*$ as in a usual two-dimensional metal. The term independent of $|\varepsilon|$ is a correction due to the tringular warping caused by the $k$-linear term, which never vanishes even in $\varepsilon \to \infty$. In the low energy region $|\varepsilon| \ll \varepsilon_0$, we have

$$\sigma(\varepsilon) = \frac{e^2}{\pi^2 h} \frac{3}{4W}.$$  

The energy-independent conductivity is essentially the same as that of the monolayer graphene.

In the case of large disorder $W > 1$, the fine structure in the density of states disappears completely (see below) and therefore the imaginary part of the self-energy becomes independent of energy and the real part becomes negligible. Thus, we always have

$$\Sigma(\varepsilon + i0) \approx -i\Gamma,$$  

with

$$\Gamma = \frac{\pi}{2} W \varepsilon_0,$$  

giving

$$\rho(\varepsilon) \approx \rho_\infty.$$  

We can calculate the conductivity (32) by putting high energy expansions (13) and (16), and obtain

$$\sigma(\varepsilon) \approx \frac{e^2}{\pi^2 h} [S(\varepsilon) + 1] + \frac{e^2}{\pi^2 h} \frac{1}{W},$$  

with

$$S(\varepsilon) = \left(\frac{\varepsilon}{\Gamma} + \frac{\Gamma}{\varepsilon}\right) \arctan \frac{\varepsilon}{\Gamma},$$  

where terms of the order of $O(1/W^2)$ are neglected. The first term of (42) comes from the $k$-square term in the dispersion and the second term is a correction due to the $k$-linear term.

The conductivity goes to the Boltzmann limit (37) for $\varepsilon > \Gamma$, while for $\varepsilon < \Gamma$ it becomes

$$\sigma = \frac{e^2}{\pi^2 h} \left(2 + \frac{1}{W}\right).$$  

It is interesting that the conductivity becomes universal, i.e., $\sigma \to 2e^2/\pi^2 h$, and never vanishes in the limit of the large disorder. At a rough estimate, we can derive this expression by putting the uncertainty relation $\varepsilon \sim \Gamma$ in (37). In Sec. VI, this universal conductivity will be revisited in terms of Einstein’s relation.
The self-consistent equation (30) can be solved easily by a numerical iteration. Figure 3 shows some examples of the density of states. We notice that the logarithmic divergence present in $\rho_0(\varepsilon)$ at $\varepsilon/\varepsilon_0 = 1/4$ is smeared out very easily and the structure in the vicinity of $\varepsilon = 0$ due to the $k$-linear term in the Hamiltonian disappears around $W \sim 0.1$ and is almost unrecognizable already for $W = 0.4$.

The corresponding plot for the conductivity is shown in Figure 4 along with the Boltzmann limit. In the weak disorder $W \ll 1$ we observe a sharp dip at zero energy. This is analog of the monolayer graphene, showing that $k$-linear dispersion around zero energy remains intact in a small disorder. Further discussion for the asymptotic value at $\varepsilon = 0$ will be given in Sec. VI. The Boltzmann conductivity drops to zero at $|\varepsilon| = (1/4)\varepsilon_0$, as the velocity vanishes at the saddle points in the dispersion, but this singularity disappears due to the finite density of states. Apart from this difference the results are almost the same as those in the Boltzmann limit. The conductivities for the large $W$’s are plotted in a different scale in the lower panel in Fig. 4, compared with the high energy expression of the Boltzmann limit (37) and the approximate expression (42). The curves deviate from the Boltzmann limit in the region $\varepsilon < \Gamma$ and never fall below $2e^2/(\pi^2h)$ even in a strong disorder. The analytic expression becomes valid already for $W \sim 1/2$.

V. LONG RANGE SCATTERERS

For the long-range disorder, we consider the scatterers with the density $n_1$ and the mean square amplitude $\langle |u_1|^2 \rangle = u^2$. Two valleys $K$ and $K'$ are now decoupled and the conductivity is written as the summation of their individual contributions. The self-energy is given by

$$\Sigma(\varepsilon) = \frac{n_1 u^2}{2\Omega} \sum_{ks} G_{jks}(\varepsilon),$$

(45)

which differs from (29) for short-range disorder in that the prefactor is larger by a factor of $2$ and that the summation is taken only in one valley. However those two elements cancel and the self-energy becomes identical with (30) in this notation. We can omit the valley index $j$ completely.

The conductivity without the vertex correction, denoted by $\sigma^0$, is equivalent with (32). The vertex part (26) is given by

$$\langle jks' | \tilde{v}_s(\varepsilon, \varepsilon') | jks \rangle = \langle jks' | v_s | jks \rangle + \frac{\hbar \omega_0}{4m^*} B(\varepsilon, \varepsilon') (se^{-i\theta_{jk}} + s'e^{i\theta_{jk}}),$$

(46)

where $\Pi$ and $B$ are dimensionless quantities defined by

$$\Pi(\varepsilon, \varepsilon') = \frac{n_1 u^2}{4\Omega} \sum_k \sum_{ss'} G_{jks}(\varepsilon) G_{jk's'}(\varepsilon'),$$

(47)

and

$$B(\varepsilon, \varepsilon') = \frac{n_1 u^2}{4\Omega} \sum_k \sum_{ss'} [1 + ss' Re(\tilde{w}_{jk}(\varepsilon^* e^{-i\theta_{jk}}))]$$

$$\times G_{jks}(\varepsilon) G_{jk's'}(\varepsilon'),$$

(48)

with $\tilde{w}_{jk} = w_{jk}(\hbar \omega_0/4m^*)^{-1}$. The conductivity correction including the valley degeneracy is then written as

$$\delta \sigma(\varepsilon) = \frac{e^2}{\pi^2 h 2W} Re \left( \frac{(B^{RA})^2}{1 - \Pi^{RA} - (B^{RR})^2} \right),$$

(49)

where $B^{RA} = B(\varepsilon + i0, \varepsilon - i0)$ and $B^{RR} = B(\varepsilon + i0, \varepsilon + i0)$. 

![Fig. 5: Calculated SCBA conductivity (solid) and Boltzmann conductivity (dotted) per spin in the case of long-range disorder, with upper and lower panel showing smaller and larger $W$’s, respectively. For the Boltzmann conductivity in the lower panel we used the expression in (52) which is valid for $\varepsilon > \varepsilon_0$. In the lower panel the approximate result given by Eq. (53) is also shown.](image-url)
In the Boltzmann limit, only the ‘RA’ part with \( s = s' \) is relevant, which can be estimated straightforwardly as

\[
\Pi^{RA} = \frac{1}{2}, \quad B^{RA} = \frac{1}{2} (1 + \text{Re}(\bar{\sigma} e^{-i\theta} \sigma)) \varepsilon. \tag{50}
\]

We can easily show \( \langle \text{Re}(\bar{\sigma} e^{-i\theta} \sigma) \rangle \varepsilon = 0 \) to obtain \( \delta\sigma/(\varepsilon^2/\pi^2) = 1/4W \) in the limit \( \varepsilon \ll \varepsilon_0 \). In the case \( \varepsilon > \varepsilon_0 \), on the other hand, \( \langle \text{Re}(\bar{\sigma} e^{-i\theta} \sigma) \rangle \varepsilon = 1 \) and therefore \( \delta\sigma/(\varepsilon^2/\pi^2) = 1/W \). The total conductivity combined with \( \sigma^0 \) in (37) or (38) becomes

\[
\sigma(\varepsilon) = \begin{cases} 
\frac{e^2}{\pi^2\hbar} \left( \frac{\varepsilon}{\varepsilon_0} + 2 \right) & (\varepsilon > \varepsilon_0), \\
\frac{e^2}{\pi^2\hbar} W & (\varepsilon \ll \varepsilon_0).
\end{cases} \tag{52}
\]

In the case of large disorder \( W > 1 \), we can derive the analytic expression similarly to the short-range case. By using the self-energy (39) we can show \( B_{RA} = 1 \), \( B_{RR} = 0 \), and \( \Pi_{RA} = 1/2 \), giving the vertex correction \( \delta\sigma(\varepsilon) = (\varepsilon^2/\pi^2)\hbar/W \) independent of energy. The total conductivity including \( \sigma^0 \) in (42) is written as

\[
\sigma(\varepsilon) = \frac{e^2}{\pi^2\hbar} S(\varepsilon) + 1 + \frac{e^2}{\pi^2\hbar} \frac{2}{W}. \tag{53}
\]

For \( \varepsilon < \Gamma \), this is reduced to

\[
\sigma = \frac{e^2}{\pi^2\hbar} \left( 2 + \frac{2}{W} \right). \tag{54}
\]

The conductivity is given by a universal value same as in the short-range case in the limit of a large disorder.

We show in Fig. 5 the conductivity numerically computed for the several \( W \)'s. The difference from the result in the short-range case is appreciable only in the clean limit \( W \ll 1 \), because the vertex correction gives only a shift by the order of \( (\varepsilon^2/\hbar)/W \). The conductivity again approaches \( 2e^2/(\pi^2) \) in the region \( \varepsilon < \Gamma \) in a strong disorder. The approximate result (54) is valid for \( W \gtrsim 0.8 \).

VI. DISCUSSION AND CONCLUSION

In Secs. IV and V we have seen that the conductivity in the case of strong disorder \( W \gg 1 \) largely deviates from the Boltzmann conductivity and converges to the order of \( 2e^2/\pi^2\hbar \). This can be understood in terms of Einstein’s relation

\[
\sigma = e^2 \rho D^*, \tag{55}
\]

where \( \rho \) is the density of states and \( D^* \) is the diffusion constant. The diffusion constant is written as \( D^* = \langle v_F^2 \rangle \tau \), where \( \langle v_F^2 \rangle \) is the average of the squared velocity over states at the Fermi energy and \( \tau \) is a relaxation time related to \( \Gamma \) through the uncertainty relation \( \Gamma = \hbar/2\tau \).

If we neglect a \( k \)-linear term in the dispersion, we have \( \langle v_F^2 \rangle = \varepsilon/m^* \) and therefore \( \sigma = n_e e^2 \tau/m^* \), with \( n_e = \rho_\infty \varepsilon \). This is nothing but the Boltzmann conductivity for \( |\varepsilon| \gg \Gamma \). In the energy range \( |\varepsilon| < \Gamma \), however, we have \( \langle v_F^2 \rangle = \Gamma/m^* \) because we take an average over the states \( \varepsilon_n \lesssim \Gamma \). Thus, the diffusion constant becomes \( D^* \sim \hbar/2m^* \) independent of energy using \( \Gamma \tau \sim \hbar/2 \).

Upon using \( \rho = \rho_\infty = m^* / \hbar^2 \), the conductivity becomes \( \sigma \sim e^2 / h \). This conductivity is universal and independent of the band parameters and the strength of scattering. It is also independent of short- or long-range disorder.

The situation becomes different in the weak disorder limit \( W \ll 1 \), where \( \sigma \) at zero energy drops from the Boltzmann conductivity almost by the factor \( W \). This behavior is essentially equivalent to that in a monolayer graphene, where the conductivity drops from the Boltzmann to the universal value \( e^2/\pi^2\hbar \) in the vicinity of the zero energy and the near-singular drop was ascribed to a reduction of the effective density of states contributing to the conductivity.\(^8\) The universal value was shown to be unaffected by magnetic fields,\(^8\) and similar near-singular behavior was shown to be present in various transport quantities.\(^{13-15}\) A universal conductivity at zero energy was also reported in the square tight-binding lattice model with one-half flux,\(^{16}\) which has \( k \)-linear dispersion as well.

This zero-energy conductivity can be explicitly estimated from (32) with the density of states (17) and the square velocity (14). The integral turns out to be independent of the imaginary part of the self-energy and returns a universal value

\[
\sigma(0) = \frac{6e^2}{\pi^2\hbar}. \tag{56}
\]

which is 6 times as large as the conductivity in the monolayer. The extra factor comes from the product of the density of states and the square velocity, \( 3|\varepsilon|/(\pi\hbar^2) \), which is larger than in the monolayer by a factor of 6 due to the existence of the satellite elliptic Fermi pockets.

Recently transport properties of a bilayer graphene were studied experimentally.\(^9\) The resistivity exhibits a prominent peak at \( \varepsilon \approx 0 \) and decreases rapidly with the increase of the energy or the electron concentration. This dependence is explained by Eqs. (42) or (53) (or the lower panel of Figs. 4 or 5) qualitatively quite well for the disorder parameter, \( 1 \lesssim W \lesssim 2 \), in the region \( n_e \lesssim 2 \times 10^{12} \text{ cm}^{-2} \). In particular, the observed peak resistivity \( \sim 6.5 \text{ k}\Omega \) corresponds well to the present result for \( W \sim 2 \) in the case of long-range disorder (dominant usually) and \( W \sim 1 \) in the case of short-range disorder. However, the observed resistivity seems to decrease much faster than that given by Eqs. (42) or (53) with a constant \( W \) for larger electron concentrations \( n_e \gtrsim 2 \times 10^{12} \text{ cm}^{-2} \). This might suggest that the effective range of the scattering potential can be comparable to the electron wavelength at these electron concentrations.

We have studied the quantum transport in the bilayer graphene in zero magnetic field with the self-consistent
Born approximation. The coexistence of the $k$-linear and $k$-square dispersion is observed as two different behaviors in the Boltzmann conductivity, as in a usual two-dimensional metal and in a monolayer graphene with a massless Dirac spectrum, which are energetically separated. The conductivity in SCBA deviates from the Boltzmann limit in the case of the strong disorder $W \gtrsim 1$, and converges to $\sim \frac{2e^2}{(\pi^2\hbar)}$ (per spin). The conductivity in the long range scatterers exhibits a qualitatively similar behavior to that in the short-range case, while the vertex correction gives a positive shift by the order of $1/W$.

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