The magnetococonductivity, which is the correction to the diagonal conductivity proportional to the second power of magnetic-field strength, is calculated in bilayer graphene in the presence of scatterers with long-range potential such as a Gaussian potential and screened Coulomb potential in a self-consistent Born approximation. The magnetoresistivity in a usual Hall bar geometry exhibits prominent double-peak structure in the vicinity of zero energy and remains very small in other regions because of the cancellation with a counter term due to the Hall effect. In a constant broadening approximation, on the other hand, the magnetoresistivity becomes negative and has a sharp double-dip structure. These features are quite similar to those in monolayer graphene.

Keywords: impurity scattering, level broadening, magnetoresistivity, Hall effect, self-consistent Born approximation, magnetoconductivity

I. INTRODUCTION

Atomically thin graphenes are attracting much attention because of their intriguing electronic properties, as has been discussed in several reviews.\textsuperscript{1-6} The bilayer graphene consists of two graphene layers stacked in the way the same as in bulk graphite. Various theoretical studies were reported on transport in different approximations.\textsuperscript{7-14} The purpose of this paper is to calculate the weak-field magnetococonductivity, correction to the conductivity proportional to $B^2$ with magnetic field $B$, in bilayer graphene within a self-consistent Born approximation, using a formula derived previously.\textsuperscript{15}

In monolayer graphene conduction and valence bands with linear dispersion cross at the K and K’ points located at a Brillouin-zone corner.\textsuperscript{16,17} In bilayer graphene, this band structure is modified by interlayer interactions. It has a zero-gap structure with quadratic dispersion, leading to nonzero density of states at zero energy.\textsuperscript{7,18-21} This is quite different from monolayer graphene having linear dispersion, where the weak-field magnetoresistivity has been predicted to exhibit a prominent double-peak structure in the vicinity of zero energy.\textsuperscript{22} It is an important question whether this double-peak feature is unique in systems with linear dispersion. It will be shown that the magnetoresistivity exhibits a similar double-peak structure also in bilayer graphene.

The paper is organized as follows: In Sect. II, following a brief review on the electronic states, the method of calculating the magnetococonductivity in a constant broadening approximation and in a self-consistent Born approximation is discussed. In Sect. III, results of numerical calculations are presented. A brief discussion and short summary are given in Sect. IV.

II. FORMULATION

In a symmetric bilayer graphene in the absence of perpendicular electric field, the electronic states near the K point are described by the $\mathbf{k} \cdot \mathbf{p}$ equation,\textsuperscript{7,18-21}

$$\mathcal{H}_0(\mathbf{k}) \mathbf{F}(\mathbf{r}) = \varepsilon \mathbf{F}(\mathbf{r}),$$

$$\mathcal{H}_0(\mathbf{k}) = \begin{pmatrix} 0 & \gamma \hat{k}_- & 0 & 0 \\ \gamma \hat{k}_+ & 0 & \gamma_1 & 0 \\ 0 & \gamma_1 & 0 & \gamma \hat{k}_- \\ 0 & 0 & \gamma \hat{k}_+ & 0 \end{pmatrix},$$

$$\hat{k}_\pm = \hat{k}_x \pm i \hat{k}_y,$$

where $\gamma$ is a band parameter, $\hat{k} = (\hat{k}_x, \hat{k}_y) = -i \nabla$, and $\gamma_1$ represents the inter-layer coupling. The parameter $\gamma$ is related to hopping parameter $\gamma_0$ through $\gamma = (\sqrt{3}/2) a \gamma_0$, where $a$ is the lattice constant given by $a = 2.46$ Å, $\gamma_0 \approx 3.16$ eV,\textsuperscript{30} and $\gamma_1 \approx 0.39$ eV.\textsuperscript{31} For states in the vicinity of the K’ point, $\hat{k}_+$ and $\hat{k}_-$ should be exchanged, but the contribution to the conductivities is the same except in the case of the valley Hall conductivity.\textsuperscript{14,32,33} The wave function with wave vector $\mathbf{k}$ is written as

$$\mathbf{F}_k(r) = \frac{1}{L} \exp(i \mathbf{k} \cdot \mathbf{r}) U(\theta_k) \mathbf{F}_k,$$

where $L^2$ is the area of the system, $k = |\mathbf{k}|$, $k_x = k \cos \theta_k$, $k_y = k \sin \theta_k$, and

$$U(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i \theta} & 0 & 0 \\ 0 & 0 & e^{i \theta} & 0 \\ 0 & 0 & 0 & e^{2i \theta} \end{pmatrix}.$$  

Then, four-component vector $\mathbf{F}_k$ satisfies

$$\mathcal{H}_0(\mathbf{k}) \mathbf{F}_k = \varepsilon \mathbf{F}_k,$$

with

$$\mathcal{H}_0(\mathbf{k}) = \begin{pmatrix} 0 & \gamma k & 0 & 0 \\ \gamma k & 0 & \gamma_1 & 0 \\ 0 & \gamma_1 & 0 & \gamma k \\ 0 & 0 & \gamma k & 0 \end{pmatrix}.$$  

The bi-
The energy bands become
\[
\varepsilon_{\pm 1}(k) = \pm \sqrt{(\gamma k)^2 + \left(\frac{\gamma_1}{2}\right)^2 - \frac{\gamma_1}{2}},
\]
\[
\varepsilon_{\pm 2}(k) = \pm \sqrt{(\gamma k)^2 + \left(\frac{\gamma_1}{2}\right)^2 + \frac{\gamma_1}{2}}.
\]

The band \(\varepsilon_{+1}(k)\) represents the lowest conduction band which touches the highest valence band \(\varepsilon_{-1}(k)\) at \(k=0\).

The bands \(\varepsilon_{\pm 2}(k)\) are the excited conduction and valence bands and \(\varepsilon_{+2}(k) - \varepsilon_{+1}(k) = \varepsilon_{-1}(k) - \varepsilon_{-2}(k) = \gamma_1\) independent of \(k\).

The density of states becomes
\[
D(\varepsilon) = \frac{g_s g_v}{2\pi^2} \left[ \frac{\gamma_1}{2} + |\varepsilon| + \theta(|\varepsilon| - \gamma_1) \left(|\varepsilon| - \frac{\gamma_1}{2}\right) \right],
\]

where \(\theta(t) = 1\) for \(t > 0\) and \(\theta(t) = 0\) for \(t < 0\), \(g_s (= 2)\) is the spin degeneracy, and \(g_v (= 2)\) is the valley degeneracy corresponding to the K and K' valleys. In the vicinity of \(\varepsilon = 0\), the Hamiltonian is reduced to a (2,2) form\(^7,18-21\)
\[
H_0 \approx \frac{\hbar^2}{2m^*} \begin{pmatrix} 0 & \hat{k}^2 & \hat{k} & 0 \\ \hat{k} & 0 & 0 & 1 \\ \hat{k} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
\[
m^* = \frac{\hbar^2}{2\gamma_2},
\]
giving \(\varepsilon_{\pm 1}(k) \approx \pm \hbar^2 k^2/2m^*\).

The density of states and the energy dispersion are shown in Fig. 1.

The velocity operators are given by
\[
v_x = \frac{1}{\hbar} \frac{\partial H_0}{\partial k_x} = \frac{\gamma}{\hbar} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]
\[
v_y = \frac{1}{\hbar} \frac{\partial H_0}{\partial k_y} = \frac{\gamma}{\hbar} \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.
\]

They can be written as
\[
v_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

The characteristic energy corresponding to \(B\) in monolayer graphene is \(h\omega_c = \sqrt{2} \gamma / l\), where \(l\) is the magnetic length given by \(l = \sqrt{\hbar/cB}\). In the present system, we can define another quantity \(h\omega_c = \hbar eB/(m^* c) = (\hbar \omega_B)^2 / \gamma_1\), using effective mass \(m^*\) at the band bottom, given by Eq. (12). We have \((\hbar \omega_B / \gamma_1)^2 = h\omega_c / \gamma_1\).

The formula of the magnetococonductivity \(\Delta \sigma\) proportional to \(B^2\) has been derived in a previous paper.\(^{15}\)
FIG. 2: (Color online) Some examples of numerical results in bilayer graphene obtained in the constant-broadening approximation for $\Gamma/\gamma_1 = 0.05$, 0.1, and 0.2. (a) The magnetoconductivity $-\Delta \sigma(z) \propto B^2$ (solid lines) and the weak-field Hall conductivity $\sigma_{xy}(z) \propto B$ (dotted lines). (b) The magnetoresistivity $\Delta \rho(z) \propto B^2$ (solid lines) and the zero-field conductivity $\sigma(z)$ (dotted lines). The negative magnetoresistance with prominent double-dip structure appears in the vicinity of zero energy. The results of the Hall conductivity and the zero-field conductivity are the same as previous results.$^{29}$

by extending the procedure used for the weak-field Hall effect.$^{34,35}$ It is given by Eq. (41) of Ref. 15 and described by the Feynman diagrams given by three hexagons shown in Fig. 1 of Ref. 15. It was explicitly calculated in monolayer graphene first within a constant broadening approximation$^{15}$ and then in a more elaborate self-consistent Born approximation.$^{22}$ In the following, we use the same approximations and therefore cannot discuss interference effects such as described by maximally crossed diagrams,$^{36-38}$ leading to weak localization behavior.$^{39-44}$

We consider the system containing scatterers, i.e.,

$$\mathcal{H} = \mathcal{H}_0 + V,$$

$$V = \sum_j u_j(\mathbf{r} - \mathbf{r}_j),$$

$$u_i(\mathbf{r}) = \int \frac{d\mathbf{q}}{(2\pi)^2} u_i(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}},$$

where $\mathbf{r}_j$ denotes position of impurities and $u_i(\mathbf{r})$ their potential, which is assumed to be isotropic, i.e., $u_i(\mathbf{q}) = u_i(\mathbf{q})$ with $\mathbf{q} = |\mathbf{q}|$. The matrix Green’s function averaged over every configuration of scatterers is written as

$$G(\mathbf{k}, \varepsilon) = [\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)]^{-1}. \quad \Sigma(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) \Sigma(\mathbf{k}, \varepsilon) U(\theta_\mathbf{k})^{-1}, \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}. \quad \Sigma(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) \Sigma(\mathbf{k}, \varepsilon) U(\theta_\mathbf{k})^{-1}, \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}. \quad \Sigma(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) \Sigma(\mathbf{k}, \varepsilon) U(\theta_\mathbf{k})^{-1}, \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}. \quad \Sigma(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) \Sigma(\mathbf{k}, \varepsilon) U(\theta_\mathbf{k})^{-1}, \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}. \quad \Sigma(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) \Sigma(\mathbf{k}, \varepsilon) U(\theta_\mathbf{k})^{-1}, \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}.$$

The dependence on the direction of $\mathbf{k}$ can be eliminated by setting

$$G(\mathbf{k}, \varepsilon) = U(\theta_\mathbf{k}) G(\varepsilon) U(\theta_\mathbf{k})^{-1}. \quad G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}.$$

$$G(\mathbf{k}, \varepsilon) = \left[\varepsilon - \mathcal{H}_0(\mathbf{k}) - \Sigma(\mathbf{k}, \varepsilon)\right]^{-1}.$$

and the transport quantities are calculated without including vertex corrections. This has been frequently used for the purpose of giving qualitative behavior of transport coefficients in graphene and related materials.$^{23-28}$

The procedure of actual calculations is exactly the same as in monolayer graphene previously considered,$^{15}$ except that the Green’s function is given by a $(4,4)$ matrix. In fact, the formula of the conductivities are exactly the same as Eqs. (72)–(77) of Ref. 15, except that $v_s$ ($s=0, \pm 1$) and the Green’s functions are all given by $(4,4)$ matrices.

Figure 2 shows some examples of numerical results for broadening $\Gamma/\gamma_1 = 0.05$, 0.1, and 0.2, the magnetocon-
ductivity $-\Delta \sigma(\varepsilon)$ and the Hall conductivity $\sigma_{xy}(\varepsilon)$ in (a) and the magnetoresistivity $\Delta \rho(\varepsilon)$ the zero-field conductivity $\sigma(\varepsilon)$ in (b). Both $\sigma_{xy}(\varepsilon)$ and $\sigma(\varepsilon)$ are in agreement with those reported previously.\(^{29}\) First, we notice that $\Delta \rho(\varepsilon)$ is negative and exhibits prominent double-dip structure in the vicinity of zero energy as shown in (b).

With the decrease of $\Gamma$, the dip depth becomes deeper roughly in proportion to $\Gamma^{-1}$ and the dip position approaches zero energy roughly in proportion to $\Gamma$.

This shows that the double-dip appears in the energy region $|\varepsilon|/\Gamma \lesssim 1$. In this region, $-\Delta \sigma(\varepsilon)$ monotonically increases with $|\varepsilon|$ from $\Delta \sigma = 0$ at zero energy, and $\sigma_{xy}(\varepsilon)$ varies almost linearly as a function of $\varepsilon$, i.e., $\sigma_{xy}(\varepsilon) \propto -\varepsilon$. Thus, no clear indication of this negative magnetoresistivity appears in $-\Delta \sigma(\varepsilon)$ and $\sigma_{xy}(\varepsilon)$. Figure 3 shows the magnetococonductivity $-\Delta \sigma(\varepsilon)$, the counter term $\sigma_{xy}(\varepsilon)^2/\sigma(\varepsilon)$ due to the Hall effect, and their difference $\Delta \sigma + \sigma_{xy}^2/\sigma$ in the vicinity of zero energy, obtained in the constant-broadening approximation. The difference becomes almost universal in this plot.

Previous calculations in the self-consistent Born approximation in the next section show that interband scattering tends to reduce the transport coefficients in the case of scatterers with a Gaussian potential. In the case of charged impurities, on the other hand, the increase of the screening effect reduces the scattering strength of each impurity and causes the opposite behavior.

### B. Self-Consistent Born Approximation

The procedure of actual calculations is again exactly the same as in monolayer graphene except that the Green’s functions and vertex functions are replaced by (4,4) matrices. The same is true of the magnetococonductivity $\Delta \sigma(\varepsilon)$. The equations for the self-energy and vertex functions are given by Eqs. (32) and Eqs. (35)–(38), respectively, of Ref. 22, where $V_n^2[k, k'; \Gamma(k')]$ with (4,4) matrix $\Gamma(k')$ is given by

$$V_n^2[k, k'; \Gamma(k')] = \begin{pmatrix} V_{11}^2 & V_{12}^2 & V_{13}^2 & V_{14}^2 \\ V_{21}^2 & V_{22}^2 & V_{23}^2 & V_{24}^2 \\ V_{31}^2 & V_{32}^2 & V_{33}^2 & V_{34}^2 \\ V_{41}^2 & V_{42}^2 & V_{43}^2 & V_{44}^2 \end{pmatrix},$$

where

$$V_n^2 = V^{(n)}(k, k') = \int_0^{2\pi} \frac{d\theta}{2\pi} u_n(k-k')^2 \cos(n\theta),$$

with integer $n$ and $\theta$ being the angle between $k$ and $k'$.

As in the case of monolayer graphene, the equation for some of the double vertex functions exhibits divergent behavior. Therefore, in general, we have to introduce imaginary self-energy $\pm \Delta$ in the denominator of the Green’s function and then take the limit $\Delta \to 0$. This procedure cannot be avoided for the calculation of the magnetococonductivity.

We choose formally the same expression as in the case of monolayer graphene,\(^{22}\) i.e.,

$$\Delta(\varepsilon) = -\operatorname{Im} \Sigma(\varepsilon) \frac{\sigma_0}{\sigma},$$

with small parameter $\delta (\delta > 0)$, where $\sigma$ and $\sigma_0$ are the diagonal conductivity calculated with and without vertex
Magnetoresistivity [units of \( \sigma / \gamma \)],
Magnetoconductivity [units of \( \sigma / \gamma \)],
Hall Conductivity [units of \( \sigma / \gamma \)].

FIG. 4: (Color online) Some examples of (a) weak-field magnetoconductivity \(-\Delta \sigma(\epsilon)\) and Hall conductivity \(\sigma_{xy}(\epsilon)\), and (b) magnetoresistivity \(\Delta \rho(\epsilon)\) and zero-field conductivity \(\sigma(\epsilon)\) for scatterers with a Gaussian potential in the presence of nonzero imaginary energy \(\delta\); calculated in the self-consistent Born approximation. The curves denoted by \(\delta = 0\) are the extrapolation toward \(\delta = 0\). \(d\gamma_1/\gamma = 0.5\). \(W = 0.15\).

FIG. 5: (Color online) Some examples of (a) weak-field magnetoconductivity \(-\Delta \sigma(\epsilon)\) and Hall conductivity \(\sigma_{xy}(\epsilon)\), and (b) magnetoresistivity \(\Delta \rho(\epsilon)\) and zero-field conductivity \(\sigma(\epsilon)\) for scatterers with a Gaussian potential, calculated in the self-consistent Born approximation. \(d\gamma_1/\gamma = 0.5\). \(W = 0.15, 0.2, 0.25,\) and \(0.3\). The corresponding Boltzmann results are denoted by thin lines.
corrections, respectively, and all the quantities are evaluated for $\Delta = 0$. This $\Delta$ roughly corresponds to $\delta \times (\hbar/2\tau)$ much smaller than $\delta \times (\hbar/2\tau_0)$ in the case of long-range scatterers, where $\delta$ and $\tau_0$ are the transport relaxation time and the simple relaxation time, respectively.

For actual numerical calculations, we introduce cutoff energy $\varepsilon_c$ and wave-number $k_c$ with $\varepsilon_c = \gamma k_c$. The cutoff is chosen as $\varepsilon_c \approx \gamma_0$, corresponding to the region where the linear dispersion is approximately valid in the band structure of monolayer graphene. Further, we choose $\gamma_1/\varepsilon_c = 0.1$ (roughly corresponding to $\gamma_1 = 0.39$ eV). We discretize $k$ and numerically solve the self-consistency equation and Bethe–Salpeter-type equations iteratively in a manner discussed previously.\textsuperscript{11,13,14,22,32,46–50} The actual value of cutoff $\varepsilon_c$ is irrelevant as long as it is sufficiently larger than typical energy scale $\gamma_1$ because states with higher energy do not contribute to physical quantities for scatterers with potential range larger than the lattice constant.

As a representative example, we first consider scatterers with Gaussian potential,

$$u_i(r) = \frac{u}{\pi d^2} \exp \left( -\frac{r^2}{d^2} \right),$$

with strength $u$ and range $d$. The scattering strength is characterized by dimensionless disorder parameter

$$W = \frac{n_i u^2}{4\pi g^2},$$

with $n_i$ being the concentration of scatterers in a unit area, as in previous calculations.\textsuperscript{11,13}

Figure 4 shows some examples of numerical results for $W = 0.05$ and $d\gamma_1/\gamma = 0.5$ in the presence of nonzero imaginary energy $\delta > 0$ and the results of extrapolation $\delta \to 0$. In (a) $-\Delta \sigma(\varepsilon)$ and $\sigma_{xy}(\varepsilon)$ are shown and in (b) $\Delta \rho(\varepsilon)$ and $\sigma(\varepsilon)$ are shown. These results indicate that the extrapolation of all these quantities can be made successfully. Similar extrapolation $\delta \to 0$ is possible for other parameters, although the results are not shown here.

Figure 5 shows some examples of the results of extrapolation $\delta \to 0$ for various values of $W$. The magnetoresistivity, shown by solid lines in (b), exhibits prominent double-peak structure in the vicinity of zero energy for small disorder $W = 0.05$. The double-peak structure becomes weaker with the increase of $W$. In the clean case $W = 0.05$ and also $W = 0.1$, the magnetoresistivity becomes negative with the increase of the energy and becomes slightly positive when the higher band is populated. The bottom of the excited band is shifted to the lower energy side due to quantum mechanical level repulsion caused by disorder ($\sim 0.7 \times \gamma_1$ for $W = 0.05$ and $\sim 0.55 \times \gamma_1$ for $W = 0.1$). For larger disorder $W > 0.1$ extra structure appears in the extreme vicinity of zero energy.

A comparison with $-\Delta \sigma(\varepsilon)$ and $\sigma_{xy}(\varepsilon)$ shown in Fig. 5(a) shows that the peak in $\Delta \rho(\varepsilon)$ corresponds to the energy where $-\Delta \sigma(\varepsilon)$ starts to deviate from a small value close to zero. No characteristic feature causing the double-peak structure in $\Delta \rho(\varepsilon)$ can be seen in $-\Delta \sigma(\varepsilon)$. Because $-\Delta \sigma(\varepsilon)$ consists of many terms (see Eq. (39) of Ref. 22, for example), its values may remain unreliable due to numerical accuracy in the region close to $\varepsilon = 0$, where $-\Delta \sigma(\varepsilon)$ remains small due to cancellations. Therefore, the extra structure near zero energy will not be discussed further in the following.

Figure 6 shows the magnetococonductivity $-\Delta \sigma(\varepsilon)$, the counter term $\sigma_{xy}(\varepsilon)^2/\sigma(\varepsilon)$ due to the Hall effect, and their difference $-\Delta \sigma - \sigma_{xy}/\sigma$ in the vicinity of zero energy for scatterers with a Gaussian potential, calculated in the self-consistent Born approximation. $d\gamma_1/\gamma = 0.5$. $W = 0.15, 0.2, 0.25$, and 0.3. The horizontal and vertical axes roughly correspond to those of Fig. 3.

In monolayer graphene, the Boltzmann conductivity becomes $\sigma = g_0 e^2/(4\pi^2 hW)$, when the potential range is small.\textsuperscript{51,52} This is independent of carrier concentration $n_s$ as long as the dependence of $W$ on $n_s$ is not taken into account. Experimentally, the conductivity increases almost linearly with $n_s$, showing that the effective scattering strength considerably decreases with $n_s$. Plausible scatterers are charged impurities.\textsuperscript{52,53} The situation is expected to be similar in bilayer graphene, although the
FIG. 7: (Color online) Some examples of (a) weak-field magnetoconductivity $-\Delta \sigma(\varepsilon)$ and Hall conductivity $\sigma_{xy}(\varepsilon)$, and (b) magnetoresistivity $\Delta \rho(\varepsilon)$ and zero-field conductivity $\sigma(\varepsilon)$ for charged impurities in the presence of nonzero imaginary energy $\approx \delta$, calculated in the self-consistent Born approximation. The curves denoted by $\delta = 0$ are the extrapolation toward $\delta = 0$. $n_i/n_c = 0.05$.

FIG. 8: (Color online) Some examples of (a) weak-field magnetoconductivity $-\Delta \sigma(\varepsilon)$ and Hall conductivity $\sigma_{xy}(\varepsilon)$, and (b) magnetoresistivity $\Delta \rho(\varepsilon)$ and zero-field conductivity $\sigma(\varepsilon)$ for charged impurities, calculated in the self-consistent Born approximation. $n_i/n_c = 0.02, 0.05,$ and $0.1$. The corresponding Boltzmann results are denoted by thin lines.
constant” in graphene systems. This means that charged
impurities are regarded as of short-range in such a way
that mixing with different bands is appreciable and the
simple (2,2) Hamiltonian given by Eq. (11) is insufficient.

As shown in Fig. 1, the density of states increases
roughly in proportion to the Fermi energy and therefore
the ratio $q_s/k_F$ remains almost independent of $\varepsilon_F$, where
$k_F$ is the Fermi wave number. Therefore, charged impuri-

ties are always considered to be short-range scatterers,
although the scattering strength decreases with $\varepsilon_F$, as
in monolayer graphene. 12 Obviously, the range is much
larger than the lattice constant and therefore contribu-
tions of scattering between the K ad K’ valleys are safely
neglected.

Figure 7 shows some examples of numerical results
for charged impurities in the presence of nonzero imagi-

nary energy $\delta > 0$ and the results of the extrapolation
$\delta \to 0$. Figure 8 shows some examples of extrapolated
results for various values of $n_i/n_c$. As shown in Fig. 8(b),
the magnetoresistivity again exhibits prominent double-
peak structure in the vicinity of zero energy. A compar-
ison with $-\Delta \sigma(\varepsilon)$ and $\sigma_{xy}(\varepsilon)$ shown in Fig. 8(a)
shows that the peak in $\Delta \rho(\varepsilon)$ corresponds to the energy
where $-\Delta \sigma(\varepsilon)$ remains very small, where no charac-
teristic feature causing the double-peak structure in $\Delta \rho(\varepsilon)$
can be seen. When the second band is occupied by electrons,
a small positive magnetoresistivity appears corresponding
to the multi-carrier transport, predicted in the Boltz-
mann theory.

In monolayer graphene, $\sigma$, $\sigma_{xy}$, $\Delta \sigma$, and $\Delta \rho$
become nearly universal, when the energy and length are all
scaled by the quantities determined by the impurity con-
centration, i.e., energy in units $\propto \gamma \sqrt{n_i}$, length in units
$\propto 1/\sqrt{n_i}$, etc. 22 Figure 9 shows the magnetoconductivity
$-\Delta \sigma$, the counter term $\sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$ due to the Hall
effect, and their difference $-\Delta \sigma - \sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$ shown in Fig. 8(a)
shows that the peak in $\Delta \rho(\varepsilon)$ corresponds to the energy
where $-\Delta \sigma(\varepsilon)$ remains very small, where no charac-
teristic feature causing the double-peak structure in $\Delta \rho(\varepsilon)$
can be seen. When the second band is occupied by electrons,
a small positive magnetoresistivity appears corresponding
to the multi-carrier transport, predicted in the Boltz-
mann theory.

Both $-\Delta \sigma(\varepsilon)$ and $\sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$ monotonically increase
from zero with the increase of the energy without any
characteristic structure. However, $-\Delta \sigma(\varepsilon)$ becomes
larger than $\sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$, causing the double-peak structure
in $\Delta \rho(\varepsilon)$. This situation is the same as for scatterers
with a Gaussian potential shown in Fig. 6. In Fig. 9, the
dependence of $\Delta \rho(\varepsilon)$ on $n_i$ is somewhat weaker than in
Fig. 8(b), but is far from the universal behavior in mono-
layer graphene. 22

**IV. DISCUSSION AND SUMMARY**

The magnetoconductivity $\Delta \sigma$, which is the correction
to the diagonal conductivity proportional to the second
power of magnetic-field strength, has been calculated in
a self-consistent Born approximation for scatterers with
long-range potential such as a Gaussian potential and
the screened Coulomb potential. The magnetoresistivity
$\Delta \rho$ becomes very small and nearly vanishes except in the
vicinity of zero energy due to near cancellation of $-\Delta \sigma$
and counter term $\sigma_{xy}/\sigma$ associated with the Hall effect.

The magnetoresistivity exhibits prominent double-

**FIG. 9:** (Color online) The magnetoconductivity $-\Delta \sigma(\varepsilon)$, the
counter term $\sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$ due to the Hall effect, and their dif-
ference $-\Delta \sigma - \sigma_{xy}(\varepsilon)/\sigma(\varepsilon)$ in the vicinity of zero energy for charged
impurities, calculated in the self-consistent Born approxima-
tion. $n_i/n_c=0.02, 0.05$, and 0.1.

$\mu_i(\varepsilon) = \frac{2\pi e^2}{\kappa(q + q_s)}$, \(36\)

where $\kappa$ is the static dielectric constant of the environ-
ment and $q_s$ is the screening constant, given by $q_s = \frac{2\pi e^2/\kappa D(\varepsilon_F)}{\kappa}$ at zero temperature with density of states
$D(\varepsilon_F)$ at Fermi energy $\varepsilon_F$. 54 The density of states and
the screening constant should be determined in a self-
consistent manner. The impurity concentration is mea-
sured in units of $n_i$, define by

$n_i = \frac{\varepsilon^2}{4\pi \gamma^2}$. \(37\)

For $\varepsilon_c \approx \gamma_0$, for example, $n_i \approx (2\sqrt{3} \pi \Omega_0)^{-1}$ with unit-cell
area $\Omega_0 = (\sqrt{3}/2)a^2$ ($n_i \approx 2 \times 10^{11}$ cm$^{-2}$ for $\varepsilon_c \approx \gamma_0$).

For graphene on SiO$_2$ substrate, we have $\kappa = (\kappa_{ox} + 1)/2$ with $\kappa_{ox} \approx 3.9$ being the static dielectric constant of
SiO$_2$, giving $\kappa \approx 2.5$. This value is used in the following
calculation. The typical wave number $k_0$ in the present
system can be chosen as $\kappa k_0 = \gamma_1$. Then, for $\varepsilon_F = 0$ in
the absence of disorder, using Eq. (10), we have $q_s/k_0 = 2g_s g_p/4(\varepsilon^2/\kappa \gamma)$, where $\varepsilon^2/\kappa \gamma \approx 1$ is a “fine-structure constant” in graphene systems. This means that charged
peak structure in the vicinity of zero energy in the case of both scatterers with a Gaussian potential and charged impurities. This feature is exactly the same as in monolayer graphene. For the positive magnetoresistance with double-peak structure, therefore, the linear dispersion of monolayer graphene is not necessary, but the presence of conduction and valence bands with vanishing gap is essential.

In monolayer graphene, features indicating the presence of double-peak structure appear in the magnetoconductivity itself, although the structure remains weak. In bilayer graphene, however, no such features are present in the magnetoconductivity. In fact, the double-peak appears in the energy region where both magnetoconductivity and Hall conductivity are small and is caused by small difference in the energy dependence of the magnetoconductivity and the counter term.

In the constant broadening approximation, the magnetoresistivity becomes negative and exhibits sharp double-peak structure in the vicinity of zero energy. This feature is again exactly the same as in monolayer graphene and is likely to be a manifestation of insufficiencies of the constant broadening approximation. In a Boltzmann theory, the magnetoresistivity in the presence of multiple bands is written as

\[ \Delta \rho = \frac{1}{\sigma} [\langle (\omega \tau)^2 \rangle - \langle \omega \tau \rangle^2] \geq 0, \]  
\[ \langle f \rangle \equiv \sum_j f_j \sigma_j(e) \left[ \sum_j \sigma_j(e) \right]^{-1}, \tag{39} \]

where \( j \) denotes band index, \( \omega_j \) the corresponding cyclotron frequency, \( \tau_j \) the relaxation time, and \( \sigma_j \) the conductivity. This shows that the magnetoresistivity becomes positive or remains nonnegative.

Because a transport equation (vertex corrections) is effectively solved in the self-consistent Born approximation, the resulting magnetoresistivity has a tendency to satisfy the above inequality (38). It is understandable, therefore, that the large double-peak structure near zero energy in the constant broadening approximation is corrected to the double-peak structure in the self-consistent Born approximation. However, this does not exclude the possibility of the magnetoresistivity becoming negative even in the self-consistent Born approximation when the Boltzmann transport equation predicts the vanishing magnetoresistivity.

As shown in Fig. 8 (b) for charged impurities, the magnetoresistivity obtained in the self-consistent Born approximation remains nonnegative in the whole energy region. For scatterers with a Gaussian potential, however, the magnetoresistivity can become slightly negative in the energy region higher than zero energy and lower than the bottom of the excited band, as shown in Fig. 5 (b). In this region, the potential range is comparable to the Fermi wavelength, while it remains much smaller for charged impurities. These results show that the sign of the magnetoresistivity can become positive or negative depending on kinds of scatterers in approximations beyond the Boltzmann transport theory.

The results obtained in this paper suggest that the magnetoresistance itself poses theoretically difficult problems because of the cancellation of the magnetoconductivity and the counter term due to the Hall effect. The same seems to be true of experiments. There have been various reports on experimental observation of the magnetoresistance in monolayer graphene.\textsuperscript{55-58} A large peak observed in the vicinity of zero energy\textsuperscript{55,56} has been analyzed in terms of the presence of large macroscopic inhomogeneity.\textsuperscript{59-63} Magnetoresistance was observed also in bilayer graphene in perpendicular and parallel magnetic fields,\textsuperscript{64-67} which seem to be analyzed also by inhomogeneities.

The magnetoconductivity can directly be measured using the so-called Corbino geometry, where only the diagonal conductivity is measured without influence of the Hall current. Unfortunately, however, the present calculations show that no clear feature appears in \(-\Delta \sigma\) itself near zero energy except that it is larger than the counter term due to the Hall effect.

So far, we have considered scatterers with potential range larger than the lattice constant \( a \). When the range becomes comparable to or smaller than \( a \), scattering between the K and K’ points should be included. At high temperatures, such scattering can be caused by zone-boundary phonons.\textsuperscript{68} At low temperature, strong and short-range scatterers such as lattice defects exhibit intriguing resonance behavior in monolayer graphene\textsuperscript{69} and in carbon nanotubes.\textsuperscript{70-72} Similar resonances are expected to appear in the present system. Effects of this intervalley scattering are left for a future study.

An extra band parameter corresponding to coupling between vertically neighboring atoms in different layers\textsuperscript{73} is known to cause trigonal warping and four band-touching points at zero energy in the energy scale of a few meV.\textsuperscript{7,18} Such a structure can easily be washed out by disorder.\textsuperscript{7} Calculations in the presence of trigonal warping are needed in clean systems, but left for future.

A band gap is usually opened up by the asymmetry between the top and bottom layers induced by a gate voltage controlling the carrier concentration experimentally.\textsuperscript{18,74-77} It is necessary, therefore, to include band-gap effects for the completeness of the theory. The band gap introduces another parameter characterizing the system and therefore will affect the magnetoresistivity in varieties of manner. This problem is beyond the scope of this paper and will be left for a future study. It is experimentally possible to control the carrier concentration without introducing a band gap with the use of a double-gate structure consisting of top and bottom gates.\textsuperscript{3,5}

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