Umklapp electron-electron scattering in weakly modulated two-dimensional electron systems

S. Uryu* and T. Ando

Institute for Solid State Physics, University of Tokyo
5–1–5 Kashiwanoha, Kashiwa, Chiba 277-8581, Japan

The resistivity determined by electron-electron scattering in two-dimensional electron systems with weak periodic electrostatic and/or magnetic-field modulations is studied over a wide range of potential period. In a weak modulation limit, when \( g \ll 2k_F \), where \( g \) is length of the smallest reciprocal lattice vector and \( k_F \) the Fermi wave number, energy gaps opening up at the boundary of the Brillouin zone play an essential role and the modulation-strength dependence of coefficient of quadratic temperature of the resistivity is \( C \propto |V_i| \ln(E_F/|V_i|) \) for \( g < 2k_F \) with \( |V_i| \) being the modulation strength and \( E_F \) the Fermi energy. Numerical calculations are systematically performed. The results are in good agreement with recent experiments.

I. INTRODUCTION

In semiconductor superlattices electron-electron scattering can contribute to the resistivity due to Umklapp processes which do not conserve the total electron momentum. Recently, this quadratic temperature dependence of resistivity in two-dimensional (2D) electron systems with weak one-dimensional (1D) periodic modulations was observed.\(^1\)\(^–\)\(^4\) The potential period can range from the order of 10 Å by using the cleaved edge overgrowth technique\(^5\)\(^,\)\(^6\) to the order of 0.5 μm in usual surface lateral superlattices. The purpose of this paper is to explore effects of electron-electron scattering on the resistivity in such 2D superlattices.

It is generally believed that electron-electron scattering process conserving total momentum, called normal process, does not contribute to the resistivity. The momentum conservation is destroyed by impurities, periodic potentials, and so on. It is well known that particularly in periodic potentials, scattering processes with the total momentum transfer equal to reciprocal lattice vectors, so-called Umklapp process, play important roles in the resistivity at low temperatures where the electron-phonon scattering is considerably suppressed.

It is a difficult problem to deal correctly with electron-electron scattering in calculations of transport coefficients. On the basis of the Kubo formula,\(^7\) the vanishing resistivity in a system with velocity proportional to momentum was given.\(^8\) Alternative ways leading to vanishing contribution of normal process with zero total velocity transfer are the Boltzmann transport equation.\(^9\)\(^,\)\(^10\) However, the Boltzmann transport equation cannot usually be solved exactly and some approximations have to be introduced such as a relaxation-time approximation. Because of the variational principle such solutions give an upper limit of the resistivity and therefore are quite useful. The Boltzmann approach has been successful in the quantitative estimation of the resistivity caused by electron-electron scattering in some metals.\(^11\)\(^,\)\(^12\)

The relaxation time in transport phenomena generally differs from that of quasiparticles. For example, the inverse relaxation time \( \tau^{-1} \) of quasiparticles of a Fermi liquid depends on the dimensionality of the system, i.e., \( \tau^{-1} \propto T^2 \) in the three-dimensional (3D) case and \( \tau^{-1} \propto T^2 \ln(E_F/k_B T) \) in the 2D case,\(^13\)\(^–\)\(^16\) where \( T \) is the temperature, \( E_F \) the Fermi energy, and \( k_B \) the Boltzmann constant. The resistivity is proportional to \( T^2 \) in both 3D and 2D systems.\(^17\)\(^–\)\(^19\)

This quadratic temperature dependence of the resistivity can change depending on the geometry of the Fermi surface. In organic conductors, for example, linear temperature dependence has been reported in addition to the ordinary quadratic dependence, whose origin has been suggested to be a crucial role of the Fermi-surface geometry such as nesting.\(^20\)\(^–\)\(^26\)

There are two types of periodic modulations in actual systems. One is an electrostatic modulation\(^1\) and the other is a magnetic-field modulation.\(^2\)\(^–\)\(^4\) Usually, the former is realized by a periodic array of gate electrodes and the latter by depositing magnetic materials on the surface. In the case of magnetic-field modulation, an averaged magnetic field can take zero or nonzero values and its strength can be freely controlled by varying the direction of the magnetization without changing the electron density. It is noted that magnetic-field modulations involve inevitable effective electrostatic modulations with the same period and possible higher harmonics due to deformation of lattice.\(^27\)\(^,\)\(^28\) Systems with magnetic-field modulation have been subjects of active theoretical study.\(^29\)\(^,\)\(^30\)

In experiments, the modulation amplitude can be estimated to some degree by fitting data to theoretical expressions\(^31\)\(^–\)\(^33\) of the oscillation of the magnetoresistivity called the Weiss oscillation\(^36\)\(^–\)\(^38\) which is observed in a 2D electron system with weak 1D modulations subject to a weak perpendicular magnetic field. Quantitatively, however, there can be some ambiguities in estimated modulation strength.

In this paper, we consider 2D electron systems with weak periodic modulations whose strength is much smaller than the Fermi energy. In usual systems fabricated on a GaAs/Al\(_x\)Ga\(_{1–x}\)As heterostructure with high mobility, energy broadening due to potential disorder is smaller than dominant energy gaps due to periodic modulations. However, the transport is dominated by usual impurity scattering and the contribution of electron-electron scattering amounts to only several percent of the residual resistivity at temperatures \( T \lesssim 20 \) K where the \( T^2 \) resistivity increase has been observed.\(^3\)\(^,\)\(^4\) The purpose of this

\(*\) Corresponding author. E-mail address: s-uryu@issp.u-tokyo.ac.jp
paper is to clarify effects of electron-electron scattering on the resistivity in such 2D systems.

This paper is organized as follows: In Sec. II, the model and method are described. In Sec. III, numerical results are given. In Sec. IV, comparison will be made to actual experiments. Finally, a summary is given in Sec. V.

II. MODEL AND METHOD
A. Model
We consider 2D electron systems with weak periodic modulations and take into account effects of the Coulomb interaction in the lowest order perturbation. Two kinds of periodic modulations, that is, electrostatic and magnetic-field modulations are considered. The Hamiltonian of the noninteracting electron system is given by

\[ \mathcal{H} = \frac{1}{2m^*} (p + eA)^2 + V_{el}, \]

where \( m^* \) is an effective mass, \( p \) a momentum, \( A \) a vector potential, and \( V_{el} \) an electrostatic potential. In the followings, we consider a magnetic-field modulation with zero average and choose the \( xy \) plane as the 2D plane.

Figure 1 illustrates a 2D system with a magnetic-field modulation. The magnetic-field modulation is usually accompanied by an inevitable electrostatic modulation with the same period mainly due to lattice distortion. Therefore, in the case of weak modulation \( B_z = B_0 \cos(g \cdot r + \delta) \) with \( g \) being the smallest reciprocal lattice vector and \( \delta \) being the phase difference between electrostatic and magnetic-field modulations, the vector and electrostatic potentials are given by

\[ A = \left( 0, \frac{B_0}{g} \sin(g \cdot r + \delta) \right), \]

\[ V_{el} = V_0 \cos(g \cdot r), \]

where \( B_0 \) and \( V_0 \) have been chosen to be positive. Then, the wave function for a noninteracting system is written as

\[ \Psi_k(r) = \frac{1}{\sqrt{\Omega}} \sum_{G} C_{G}(k) e^{i(k+G) \cdot r}, \]

where \( \Omega \) is the system area and \( G = ng \) with \( n = 0, \pm 1, \ldots \). The state specified by a wave vector \( k \) is represented as superposition of plain waves with the wave vector \( k + G \).

The Schrödinger equation is represented as

\[ \sum_{G'} \mathcal{H}_{G,G'} C_{G'}(k) = EC_{G}(k), \]

where \( C_{G}(k) \) and \( C_{G'}(k) \) are the coefficients in Eq. (2.3), \( E \) the eigen energy, and the matrix elements \( \mathcal{H}_{G,G'} \) are

\[ \mathcal{H}_{G,G} = \frac{\hbar^2}{2m^*} (k + G)^2 + \frac{e^2 B_0^2}{4m^* g^2}, \]

\[ \mathcal{H}_{G,G \pm 2G} = V_{el} = V_0 \frac{g}{2} \pm i\frac{e B_0 k_y}{2m^* g} e^{\pm id}, \]

\[ \mathcal{H}_{G,G \pm 2G} = -\frac{e^2 B_0^2}{8m^* g^2}. \]

The other elements are zero and \( g \) is assumed to be directed along the \( x \) axis, i.e., \( g = (g,0,0) \). Using the coefficient \( C_{G}(k) \), the group velocity is given by

\[ v_x = \frac{\hbar k_x}{m^*} + \frac{\hbar}{m^*} \sum_{G} G_x |C_{G}(k)|^2, \]

\[ v_y = \frac{\hbar k_y}{m^*} + \frac{e B_0}{m^* g} \sum_{G} \text{Im}[C_{G}(k)C_{G}^* (G-g)(k)]. \]

It is noted that the group-velocity change perpendicular to the modulation direction is exactly zero in the absence of a magnetic-field modulation. Energy bands and wave functions are calculated numerically on the basis of Eqs. (2.4)–(2.6) by taking adequate number of plane waves.

B. Energy bands
In a weak modulation limit, a gap \( 2|V_{el}(k_y)| \) opens up at \( k_x = \pm g/2 \). Consider first the case where either an electrostatic or magnetic-field modulation is present. In the case of electrostatic modulation \( V_{el} = V_0/2 \) or \( |V_{el}| \equiv |V_{el}^\pm| = V_0/2 \) independent of \( k_y \), while in the case of magnetic-field modulation \( V_{el}^\pm = \mp i\omega_c (k_y/2g)e^{\pm id} \) or \( |V_{el}| \equiv |V_{el}^\pm| = h\omega_c (k_y/2g) \) with

\[ \omega_c = \frac{eB_0}{m^*}. \]

In the latter case where \( V_{el} \propto k_y \), the effective potential changes its signature at \( k_y = 0 \) and no periodic potential appears for states with \( k_y = 0 \). In both cases, the energy bands have mirror symmetry around the \( k_x \) and \( k_y \) axes.

To the first order of modulation strength, the wave function is written as

\[ \Psi_k(r) = \frac{e^{ik \cdot r}}{\sqrt{\Omega}} [C_0(k) + C_g(k)e^{ig \cdot r} + C_{-g}(k)e^{-ig \cdot r}]. \]

The expanding coefficient \( C_0(k) \) is always of the order of unity. The coefficient \( C_{\pm g}(k) \) is of the first order of \( |V_{el}| \) except in the energy-gap region in the vicinity of \( k_x = \mp g/2 \). In the case of the electrostatic modulation, the expansion coefficients have the symmetry

\[ C_{\pm g}(k_x, k_y) = C_{\pm g}(k_x, -k_y), \]

and in the case of the magnetic-field modulation

\[ C_{\pm g}(k_x, k_y) = -C_{\pm g}(k_x, -k_y), \]

because \( V_{el} \propto k_y \). In both cases the following relation is satisfied:

\[ C_g(k) = C_{-g}(-k). \]

When \( g < 2k_F \) with \( k_F \) being the Fermi wave vector, the Fermi surface crosses these gaps at \( k_x = \pm g/2 \) and \( k_y = \pm k_g \) with

\[ k_g = \sqrt{k_F^2 - g^2}/4. \]

The wave vectors corresponding to each gap are denoted as \( k_c, -k_c, -k_c, \) and \( -k_c \) where

\[ k_c = (g/2, -k_y), \]

\[ k_c = (g/2, k_y). \]

In the vicinity of these points there is a gap-region where
the energy dispersion deviates significantly from $\hbar^2k^2/2m^*$. The width is given by

$$\Delta k_x = \frac{k_F^2}{g} \frac{|V_{\pm}|}{E_F}, \quad \Delta k_y = \frac{k_F^2}{2k_y} \frac{|V_{\pm}|}{E_F},$$

(2.14)

which give

$$\Delta k = \frac{k_F^2}{gk_y} \frac{|V_{\pm}|}{E_F},$$

(2.15)

along the Fermi surface.

When the wave vector $k$ is swept along the Fermi surface, the phase of $C_{\pm k}(k)$ changes by $\pi$ whenever $k$ crosses a line of $k_z = \mp g/2$. For example, we have in the case of electrostatic modulation with the use of Eq. (2.9)

$$C_{-g}(g/2+0, k_y) = -C_{-g}(g/2-0, k_y)$$

(2.16)

$$= C_{-g}(g/2-0, -k_y) = C_{-g}(g/2+0, -k_y),$$

while in the case of magnetic-field modulation with the use of Eq. (2.10)

$$C_{-g}(g/2+0, k_y) = -C_{-g}(g/2-0, k_y)$$

(2.17)

$$= C_{-g}(g/2-0, -k_y) = C_{-g}(g/2+0, -k_y).$$

When the modulation strength increases, energy gaps open up at $k_x = ng/2$ with $n = \pm 1, \pm 2, \cdots$ with width proportional to $|V_i|/|E_F|$. The parameter $m^*|V_i|/\hbar^2g^2$ characterizes the strength of effects of energy gaps. In fact, when $m^*|V_i|/\hbar^2g^2 \approx 1$, $|V_i|$ becomes comparable to the energy difference between states with $k$ and those with $k\pm 2g$ at $k = \mp g/2$. Further, the extension of energy-gap region $\Delta k_x$ becomes comparable to the Brillouin-zone width $g$. This shows that effects of higher order terms become important in a system with smaller $g$.

Figure 2 shows examples of the Fermi surface for $V_0/E_F = 0.1$ and $\hbar\omega_c = 0$, i.e., $|V_i|/|E_F| = 0.05$. In Fig. 2(a) where $g/k_F = 0.8$ and $m^*|V_i|/\hbar^2g^2 \approx 0.08 \ll 1$, dominant energy gaps of the first order of modulation strength open up at $\pm k_x$ and $\pm k_y$. Figure 2(b) shows the blowup of an energy-gap region at $k_x$ where the width is about $\Delta k_x$ in the $k_x$ direction and about $\Delta k_y$ in the $k_y$ direction where $\Delta k_x$ and $\Delta k_y$ are defined in Eq. (2.14). We define ‘$+$’ branch of the Fermi surface by the dispersion with higher energy at $k_x = \pm g/2$, i.e., $|k_x| > g/2$ and ‘$-$’ branch by that with lower energy i.e., $|k_x| < g/2$ as shown in the figures. In usual actual systems with small $g$ as shown in Figs. 2(c) and (d) where $g/k_F = 0.1$ and $m^*|V_i|/\hbar^2g^2 = 5 > 1$, higher order energy gaps also open at $k_x = ng/2$ with $n = \pm 1, \pm 2, \cdots$.

The case of $g/k_F \approx 2$ is exceptional. In this case, the lowest order energy gaps open up at $k_x = k_F$ and there are no $+$ branches of the Fermi surface. Then, the energy-gap width of the Fermi surface becomes

$$\Delta k \approx \Delta k_y = k_F \sqrt{\frac{V_0}{2E_F}},$$

(2.18)

in the case of electrostatic modulation because the Fermi surface nearly touches the boundary of the first Brillouin zone. In the case of a magnetic-field modulation, the modulation strength is considerably reduced because $V_i \propto k_y$ and $k_y \sim 0$ at energy gaps. Assume the energy-gap width to be $\Delta k_y$. Then the effective gap is regarded as $|V_i| \sim \Delta k_y \hbar\omega_c/4k_F$. Setting this equal to $\hbar^2 \Delta k_y^2/2m^*$, we have

$$\Delta k \approx \Delta k_y = \frac{k_F\hbar\omega_c}{4E_F}, \quad |V_i| \approx \frac{(\hbar\omega_c)^2}{16E_F}$$

(2.19)

When both types of modulations coexist, the symmetry of the bands around the $k_z$ axis is destroyed. When $\delta = \pi/2$ corresponding to the situation shown in Fig. 1, in particular, the effective potential $V_{\pm}$ vanishes at $k_y = -g(V_0/\hbar\omega_c)$ but does not at $k_y = g(V_0/\hbar\omega_c)$. When the modulation period is much larger than the electron wavelength $\lambda_F$, this corresponds to the condition $\hbar\omega_c/V_0 = g/k_F$ because $|k_y| \approx k_F$, i.e., disappearance of the Weiss oscillation.35

C. Boltzmann transport equation

We use the Boltzmann transport equation in calculations of the resistivity

$$\frac{\partial f}{\partial t} \bigg|_{\text{field}} = -I,$$

(2.20)

where $f$ is the distribution function, the left hand side is the rate of change of $f$ due to external fields and the collision term $I$ represents that due to scattering events. We consider the case where the external field is only an electric field in the following. Expanding the distribution function $f$ around the Fermi-Dirac distribution function $f^0$, the distribution function is

$$f = f^0 + \Phi \frac{\partial f^0}{\partial E},$$

(2.21)

where $\Phi$ is in proportion to an electric field. Linearizing with respect to an electric field, the field driven term in Eq. (2.20) is written as

$$\Phi \frac{\partial f^0}{\partial E},$$

(2.22)

where $\Phi$ is the group velocity of state with $k$ and $E$ the electric field vector. The electron-electron scattering part in the collision term of Eq. (2.20) is given by

$$I_{ee} = \frac{2}{k_B T} \sum_{k_1, k_2, k_3, k_4} \left( \Phi_1 + \Phi_2 - \Phi_3 - \Phi_4 \right) f_{01}^0 f_{12}^0 (1-f_{02}^0) (1-f_{03}^0) \times W(k_1, k_2; k_3, k_4),$$

(2.23)

where $f^0_0 = f^0_0(k_i)$ and $W(k_1, k_2; k_3, k_4)$ is the transition probability of two-electron scattering process from initial states $k_1$ and $k_2$ to final states $k_3$ and $k_4$ and will be written as $W$ in the following for simplicity.

We consider the case of dominant impurity scattering where electron-electron scattering constitutes only a small correction to the total scattering. Because the impurity scattering occurs almost equally for states with $k$ lying on the Fermi line, we can approximately determine the distribution function variationally assuming a
relaxation time $\tau_0$ independent of the $k$ direction, i.e.,
\[ \Phi = e v \cdot E \tau_2. \] (2.24)

Minimization of the resistivity gives
\[ \frac{1}{\tau} = \frac{1}{\tau_0} + \frac{1}{\tau_{ee}} \] (2.25)
where $\tau_0$ and $\tau_{ee}$ are relaxation times due to impurity scattering and electron-electron scattering, respectively. The relaxation time due to electron-electron scattering is given by
\[ \frac{1}{\tau_{ee}} = \frac{1}{k_B T} \sum_{k_1, k_2, k_3, k_4} (\Delta v \cdot u)^2 f_1^0 f_2^0 (1 - f_3^0) (1 - f_4^0) W \times \left[ 2 \sum_k (v \cdot u)^2 \left( -\frac{\partial f_1^0}{\partial E} \right) \right]^{-1}, \] (2.26)
where
\[ \Delta v = v_3 + v_4 - v_1 - v_2, \] (2.27)
and $u$ is the unit vector in the direction of electric field. Then, the resistivity due to electron-electron scattering $\Delta \rho$ is given by
\[ \Delta \rho = \frac{\Omega}{k_B T} \sum_{k_1, k_2, k_3, k_4} (\Delta v \cdot u)^2 f_1^0 f_2^0 (1 - f_3^0) (1 - f_4^0) W \times \left[ 2 \sum_k (v \cdot u)^2 \left( -\frac{\partial f_1^0}{\partial E} \right) \right]^{-2}. \] (2.28)

It should be noted that Eqs. (2.25) and (2.26) are valid in the case of dominant impurity scattering, i.e., $\tau_0 \ll \tau_{ee}$. In this case the relaxation is nearly isotropic along the Fermi line and therefore $\tau_{ee}$ independent of the $k$ direction is semi-quantitatively valid.\(^3\) In the case of dominant electron-electron scattering, the Umklapp electron-electron scattering depends strongly on $k$ along the Fermi line as will be shown in the following and the present approximation of a $k$ independent relaxation time can become poor.\(^4\)

The transition probability is calculated with the use of the Fermi’s golden rule. The symmetry of the spatial part of the wave function with respect to exchange of two particles is taken into account by using averaged transition probability over symmetric and antisymmetric two-particle states, that is,\(^1\)\(^,\)\(^1\)
\[ W(k_1, k_2; k_3, k_4) = \sum_{\Delta G} W_{\Delta G} \delta_{\Delta k} \delta_{\Delta G} \delta(\Delta E), \] (2.29)
\[ W_{\Delta G} = \frac{2\pi \hbar^4}{\hbar m^* \tau} |A|^2 + |B|^2 + |A - B|^2, \]
where
\[ \Delta G = G_3 + G_4 - G_1 - G_2, \]
\[ \Delta k = k_3 + k_4 - k_1 - k_2, \] (2.30)
\[ \Delta E = E_3 + E_4 - E_1 - E_2, \]
with $E_i$ being the energy of state of $k_i$ ($i = 1, \ldots, 4$). The term $A$ is given by
\[ A = \sum_{G_1, G_2, G_3, G_4} \bar{V}_{ee}(q') C_{G_1}(k_4) C_{G_2}(k_3) C_{G_3}(k_1) C_{G_4}(k_2), \] (2.31)
where
\[ \bar{V}_{ee}(q') = \frac{k_F}{2 \pi \epsilon e^2} V_{ee}(q'), \] (2.32)
with $V_{ee}(q')$ being the effective Coulomb interaction given later in Eq. (2.41) and
\[ q' = k_3 - k_1 + G_3 - G_1 = k_2 - k_4 + G_2 - G_4. \] (2.33)
The summation in Eq. (2.31) is taken over sets of $G_1, G_2, G_3, G_4$ which satisfy $\Delta G = nG$ with $n = 0, \pm 1, \cdots$. Therefore, the total momentum transfer $h\Delta k$ is given by
\[ \Delta k + \Delta G = 0. \] (2.34)
The term $B$ is obtained by setting
\[ q' = k_4 - k_1 + G_4 - G_1 = k_2 - k_3 + G_2 - G_3, \] (2.35)
in the expression of $A$.

The term $|A|^2$ describes scattering processes between different spins such as $(k_1^\uparrow, k_2^\uparrow) \rightarrow (k_3^\uparrow, k_4^\uparrow)$ and $|B|^2$ $(k_1^\uparrow, k_2^\uparrow) \rightarrow (k_3^\downarrow, k_4^\downarrow)$. The term $|A - B|^2$ describes scattering between electrons with same spin such as $(k_1^\uparrow, k_2^\uparrow) \rightarrow (k_3^\uparrow, k_4^\uparrow)$ and its exchange counter part $(k_1^\downarrow, k_2^\downarrow) \rightarrow (k_3^\uparrow, k_4^\downarrow)$.

At a low temperature, the numerator of Eq. (2.28) can be written as
\[ \sum_{k_1, k_2, k_3, k_4} (\Delta v \cdot u)^2 f_1^0 f_2^0 (1 - f_3^0) (1 - f_4^0) W \approx \int dE_1 dE_2 dE_3 dE_4 f_1^0 f_2^0 (1 - f_3^0) (1 - f_4^0) \delta(\Delta E) \times \sum_{\Delta G} \int d|k_1| d|k_2| d|k_3| d|k_4| (\Delta v \cdot u)^2 W_{\Delta G} \delta_{\Delta k} \delta_{\Delta G} \times \delta(E_1 - E_F) \delta(E_2 - E_F) \delta(E_3 - E_F) \delta(E_4 - E_F) \] (2.36)
where the velocity is represented by that on the Fermi surface and the summation is replaced by the integration. Equation (2.36) is rewritten as
\[ \int dE_1 dE_2 dE_3 dE_4 f_1^0 f_2^0 f_3^0 f_4^0 \delta(\Delta E) = \frac{(k_B T)^3}{2} \int_{-\infty}^{\infty} df^0 (1 - f^0)^2 (\pi^2 + \epsilon^2) \]
\[ = \frac{2\pi^2(k_B T)^3}{3}, \] (2.37)
where $\epsilon = (E - \mu)/k_B T$ and $\mu$ is the chemical potential and
\[ \int d|k_1| d|k_2| d|k_3| d|k_4| (\Delta v \cdot u)^2 W_{\Delta G} \delta_{\Delta k} \delta_{\Delta G} \times \delta(E_1 - E_F) \delta(E_2 - E_F) \delta(E_3 - E_F) \delta(E_4 - E_F) \]
\[ = \frac{1}{\hbar^4} \int dq \frac{(\Delta v \cdot u)^2 W_{\Delta G}}{|v_1| |v_3| |v_2| |v_4|}, \] (2.38)
where $q$ is defined by $q = k_3 - k_1 = k_2 - k_4 - \Delta \bf{G}$. When $q$ is given, $k_1, k_2, k_3,$ and $k_4$ are determined, but there
can be several sets. Therefore, the integral in Eq. (2.38) includes summation over possible sets of \( \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \) and \( \mathbf{k}_4. \)

Then, the resistivity becomes

\[
\Delta \rho = \tilde{C} \left( \frac{\lambda_F E_F}{\varepsilon} \right)^2 \int \frac{k_B T}{E_F} \frac{2 \hbar}{c} \int dq \frac{(|\mathbf{\Delta v} \cdot \mathbf{u}|)^2 W_{\Delta G}}{(|\mathbf{v}_1 \times \mathbf{v}_3|)(|\mathbf{v}_2 \times \mathbf{v}_4|)},
\]

(2.39)

where \( \lambda_F \) is the Fermi wave length and \( \tilde{C} \) a dimensionless coefficient giving the resistivity.

The integrand consists of three factors such as the element of total velocity transfer along the electric field \( \mathbf{\Delta v} \cdot \mathbf{u} \), the transition probability \( W_{\Delta G} \), and the density of states \( (|\mathbf{v}_1 \times \mathbf{v}_3|)(|\mathbf{v}_2 \times \mathbf{v}_4|) \). The resistivity is represented by the summation over \( \Delta G \), i.e., the contribution of normal processes with \( \Delta G = 0 \) and that of Umklapp processes with \( \Delta G \neq 0 \).

Consider a circular Fermi surface and a normal process illustrated schematically in Fig. 3(a). Another possible process is shown in Fig. 3(b). The density of states \( |\mathbf{v}_2 \times \mathbf{v}_4| \) diverges when \( \mathbf{k}_3 \approx \mathbf{k}_1 \) or \( q \approx 0 \) and when \( k_3 \approx -k_1 \) or \( q \approx 2k_F \) because \( \mathbf{v}_2 \) and \( \mathbf{v}_4 \) become parallel to each other. Although the same happens to \( |\mathbf{v}_1 \times \mathbf{v}_3| \), the transition probability \( W_{\Delta G} \) does not appear if the integral is expressed in terms of the direction angle \( \theta_1 \) of \( \mathbf{k}_1 \) and \( \theta_3 \) of \( \mathbf{k}_3 \) instead of \( q \). In fact in the vicinity of \( \theta_3 \approx \theta_1 \) \( (q \approx 0) \) and \( \theta_3 \approx \theta_1 + \pi \) \( (q \approx 2k_F) \), we have

\[
\int \frac{dq}{|\mathbf{v}_1 \times \mathbf{v}_3|(|\mathbf{v}_2 \times \mathbf{v}_4|)} \propto \int d\theta_1 d\theta_3 \left\{ \begin{array}{ll}
\frac{1}{|\mathbf{v}_1 \times \mathbf{v}_3|(|\mathbf{v}_2 \times \mathbf{v}_4|)} & (\theta_3 \sim \theta_1 + \pi), \\
\frac{1}{|\mathbf{v}_1 \times \mathbf{v}_3|(|\mathbf{v}_2 \times \mathbf{v}_4|)} & (\theta_3 \sim \theta_1).
\end{array} \right.
\]

(2.40)

The integral diverges logarithmically, which is the reason that the relaxation time of electron-electron scattering is \( \sim T^2 \ln(E_F/k_BT) \) in the 2D system.

Figure 4(a) shows examples of backward scattering processes \( q \approx 2k_F \). In this case the Fermi surface can locally be regarded as consisting of two parallel lines as illustrated in Fig. 4(b). Therefore for scattering from an initial state \( \mathbf{k}_1 \) to a final state \( \mathbf{k}_3 \) there are infinite number of initial states \( \mathbf{k}_2 \) of electrons contributing to the scattering. The situation does not change even if \( q \) deviates slightly from \( 2k_F \) as long as the Fermi surface consists of two parallel lines. The same divergence occurs at \( q \approx 0 \) as illustrated in Figs. 4(c) and (d). This is the origin of the divergence of Eq. (2.40) and of the temperature dependence \( \tau_{ee} \sim T^2 \ln(E_F/k_BT) \). This divergence is irrelevant in the resistivity because of the vanishing velocity transfer \( (|\mathbf{\Delta v} \cdot \mathbf{u}|)^2 \).

In the presence of modulation, the total velocity transfer for normal processes is not exactly zero in general. In this case, however, the relation \( \mathbf{k}_1 = \mathbf{k}_3 \) and \( \mathbf{k}_2 = \mathbf{k}_4 \) holds for the forward scattering and \( \mathbf{k}_1 = \mathbf{k}_4 \) and \( \mathbf{k}_2 = \mathbf{k}_3 \) for the backward scattering. Therefore, the integral given by Eq. (2.39) never diverges because the total velocity transfer vanishes when the density of states diverges. The integral converges for Umklapp processes as will be discussed below.

In the absence of magnetic-field modulation, the resistivity \( \Delta \rho_{xy} \) vanishes due to the fact that \( v_x \propto k_x \) as shown in Eq. (2.6) leading to \( \Delta v_y = 0 \). In the presence of a weak magnetic-field modulation, \( \Delta v_y \propto B_0 \). This velocity transfer is negligibly small as compared to \( \Delta v_x \approx \hbar g/m^* \). Further, there is no divergence in the integral for \( \Delta \rho_{xy} \). Therefore, we will consider only \( \Delta \rho_{xx} \) throughout in the following.

If we take into account the screening effect, the effective Coulomb interaction is written as\(^{41}\)

\[
V_{ee}(q) = \frac{2\pi e^2 F(q)/\kappa q}{1 + 2\pi e^2 F(q)/\kappa q}.
\]

(2.41)

where \( \kappa \) is the static dielectric constant, \( F(q) \) the polarizability of free electron calculated in the random phase approximation,\(^{42}\) and \( F(q) \) a form factor given by

\[
F(q) = \int dz dz' |\xi(z)|^2 |\xi(z')|^2 e^{-q|z-z'|},
\]

(2.42)

with \( \xi(z) \) being an envelope of wave function along the \( z \) direction of thickness. As long as modulation strength is small, its effects on the polarization function can safely be neglected and a spherical Fermi surface is assumed. When \( q \) is smaller than \( 2k_F \) and then Eq. (2.41) is written as

\[
V_{ee}(q) = \frac{2\pi e^2 F(q)}{\kappa q (q + q_{TF} F(q))}.
\]

(2.43)

where \( q_{TF} \) is the Thomas-Fermi screening wave vector given by

\[
q_{TF} = \frac{2\pi m^*}{\hbar^2}.
\]

(2.44)

As a form factor, we shall use the Fang-Howard variational function\(^{13}\)

\[
\xi(z) = \sqrt{\frac{m^*}{2\pi}} \exp \left( -\frac{bz^2}{2} \right).
\]

(2.45)

D. Dominant Umklapp process

Before presenting explicit numerical results, we shall discuss dominant Umklapp processes in a weak modulation limit. When \( g > 2k_F \), there are no energy gaps on the Fermi surface and \( C_{\pm g}(\mathbf{k}) \) is always of the first order of modulation strength. Then, the lowest order Umklapp processes are those with \( \Delta G = \pm g \) and the transition probability is proportional to \( |V_{ij}|^2 \). Therefore, \( \tilde{C} \) depends quadratically on the modulation strength. When \( g > 4k_F \), Umklapp processes cannot generally contribute to the scattering because \( \Delta G = 0 \) is not satisfied for \( k_j \) \( (j = 1, \ldots, 4) \) in the first Brillouin zone except when \( \Delta G = 0 \).

In the following we shall concentrate ourselves to the case \( g < 2k_F \). In this case the Fermi surface crosses the energy-gap regions in the vicinity of \( k_x = \pm g/2 \). As the wave function is superposition of plane waves with \( \mathbf{k} \) and \( \mathbf{k} \pm \mathbf{g} \) as shown in Eq. (2.8), two-particle scattering from
the processes shown in Fig. 5(a), i.e., $k_1 \sim -k_3$ as shown in Fig. 5(b). In both cases $k_3$ is arbitrary on the Fermi surface.

In the following we shall concentrate on examples of processes shown in Fig. 5(a), i.e., $k_1$ lies on the $+$ branch, i.e., $k_{1z} > g/2$, $k_2 \sim k_3$, and $|k_{3z}| > g/2$. In this case $k_2$ lies always on the $-$ branch of the Fermi surface different from that $k_1$ lies on, i.e., $k_{2z} < g/2$. As already mentioned, it describes the process $(G_1, G_2, G_3, G_4) = (-g, 0, 0, 0)$. For the same set $k_{j}$ ($j = 1, \ldots, 4$) and $\Delta G = g$, the Coulomb matrix element $A$ in Eq. (2.31) contains a contribution of the process $(G_1, G_2, G_3, G_4) = (0, -g, 0, 0)$. This process is illustrated in Fig. 6. The former matrix element involves $C_0^0(k_1)C_0^0(k_3)C_{-g}(k_2)$ and the latter $C_0^0(k_1)C_0^0(k_3)C_0^0(k_1)C_{-g}(k_2)$.

In the case of electrostatic modulation, $C_{-g}(k_1) \sim -C_{-g}(k_2)$ because $k_1$ and $k_2$ lie on different branches of the Fermi surface [see Eq. (2.16)] and therefore these two processes tend to cancel each other. The cancellation is not exact because $q'$ is different between them.

In the case of magnetic-field modulation, on the other hand, $C_{-g}(k_1) \sim C_{-g}(k_2)$ [see Eq. (2.17)] and therefore these two processes tend to enhance each other. Therefore, the resistivity in the case of electrostatic modulation becomes smaller than that in the case of magnetic-field modulation for small $g$.

The situation becomes opposite when $|k_{3z}| < g/2$, however. In this case $k_2$ lies on the $+$ branch same as that of $k_1$ as shown in Fig. 7(a). The corresponding process $(G_1, G_2, G_3, G_4) = (0, -g, 0, 0)$ is shown in Fig. 7(b). Therefore, $C_{-g}(k_1) \sim -C_{-g}(k_2)$ for electrostatic modulation and $C_{-g}(k_1) \sim -C_{-g}(k_2)$ for magnetic-field modulation.

When $k_3$ approaches the energy gap at $k_6$ as shown in Fig. 8(a), the transition from $k_3$ to $k_4$ becomes nearly a backward scattering and the density of states $|v_2 \times v_4|^{-1}$ diverges. This singularity is present only when $k_3$ is exactly at $k_6$, and disappears when $k_1$ is moved away from $k_6$ only slightly. In fact, a slight shift of $k_1$ from $k_6$ leads to a large shift of $k_2$ outside the energy-gap region at $k_6$, as shown in Fig. 8(a). A similar divergence occurs for $k_3 \sim -k_6$ as shown in Fig. 8(b), where transition from $k_2$ to $k_4$ becomes nearly a forward scattering. A slight shift of $k_1$ from $k_6$ weakens the singularity again because it leads to a large shift of $k_2$ outside the energy-gap region at $k_6$.

Although the singularity of the integrand at $k_1 = k_6$ and $k_1 = k_6$ or $k_3 = -k_6$ does not cause any divergence in the resistivity, it dominates $\Delta \rho_{xx}$ in the weak-modulation limit $|V_1| \rightarrow 0$. Let $\theta_3$ be the direction angle of $k_3$ and $\theta_6$ of $k_6$. Then, the density of states $|v_2 \times v_4|^{-1}$ diverges like $|\theta_3 - \theta_6|^2$ at $k_6 = k_6$, but the divergence is cutoff when $k_1 \neq k_6$. Therefore, we can immediately obtain an approximate dependence of $\tilde{C}$ on $|V_1|$ as

$$\tilde{C} \propto \frac{g}{k_{1z}} \left( \ln \left( \frac{E_F}{|V_1|} \right) + c \right),$$

where $c$ is a constant. In deriving the above use has been made of $|k_1 - k_6| \sim \Delta k$ with $\Delta k$ given by Eq. (2.15) and $\Delta \rho_{xx} \sim h\rho g/m^*$. Because this $\tilde{C}$ is dominated by contributions of $k_1$ corresponding to the condition $|k_1 - k_6| \sim \Delta k$, when $k_2$ is pushed away from the energy-gap region, cancellations or enhancement depending on modulation types mentioned above become almost irrelevant.

When the nonzero thickness of the 2D system is considered, the Coulomb matrix element decreases because of the form factor $F(q') < 1$ except at $q' = 0$ for which $F(0) = 1$. In the limit of weak modulation, contributions from $q' \sim 2k_F$ and $q' \sim 0$ are dominant. When $q' \sim 0$, the term $A$ determined by $q' \sim 0$ remains same but $B$ determined by $q' \sim 2k_F$ decreases when the thickness effect is considered. Because the total probability is given by $A^2 + B^2 + (A - B)^2$ ($A$ and $B$ can be chosen as real and positive), it can be reduced or enhanced depending on $B/A$. In fact, the thickness enhances the resistivity when $B/A \ll 1$, while it reduces the resistivity when $B/A \sim 1$. In actual GaAs/Al$_x$Ga$_{1-x}$As systems, $|B/A| \propto V(2k_F)/V(0) = |1 + 2k_F/\epsilon_T F(2k_F)|^{-1}$ lies in the energy gap region between these two regimes.

In the case of $g \approx 2k_F$, there is a significant difference between the cases of electrostatic and magnetic-field modulations. An energy-gap region $|k_1 - k_6| \sim \Delta k$ becomes different between the two cases as shown in Eqs. (2.18) and (2.19). In the case of electrostatic modulation, $\Delta k$ is given by Eq. (2.18) and then $\tilde{C}$ is given by

$$\tilde{C} \propto \frac{V_0}{E_F} \left( \ln \left( \frac{E_F}{V_0} \right) + c \right).$$

The coefficient $\tilde{C}$ is enhanced as compared to that in the case of $g < 2k_F$ due to the extension of energy-gap
regions. While in the case of magnetic-field modulation, \( \Delta k \) is given by Eq. (2.19) and then \( \tilde{C} \) is given by
\[
\tilde{C} \propto \frac{\hbar v_c}{E_F} \left[ \ln \left( \frac{E_F}{\hbar v_c} \right) + c \right]. \tag{2.48}
\]
This is the same dependence as that in the case of \( g < 2k_F \) because the energy-gap region is not enlarged due to the reduction of modulation strength. The details of calculations of modulation-strength dependence are described in Appendix.

III. NUMERICAL RESULTS

At a GaAs/Al\(_x\)Ga\(_{1-x}\)As heterostructure, the effective mass is \( m^*/m \sim 0.067 \), the static dielectric constant is \( \kappa \sim 13.2 \), and \( 2\pi/q_{TF} \sim 300 \) Å. For the electron concentration \( n \sim 2 \times 10^{11} \text{ cm}^{-2} \), the Fermi wavelength is \( \lambda_F \sim 600 \) Å and the average thickness \( 3/\lambda_F \sim 100 \) Å. Therefore, we choose \( q_{TF}/k_F = 2.0 \) and \( 3/\lambda_F = 0.2 \).

A. Electrostatic modulation

Figure 9 shows the calculated coefficient \( \tilde{C} \) as a function of the modulation strength \( V_0 \) for various values of \( g \). Solid lines are the results for nonzero thickness and dotted ones for zero thickness. As shown in Fig. 9(a), \( \tilde{C} \) for small \( g \) such as \( g/k_F = 0.1 \) depends almost linearly on the modulation strength due to effects of higher order energy gaps. With the increase of \( g \) the resistivity increases, and for \( g/k_F = 1.2 \) and 1.6 shown in Fig. 9(b) the modulation-strength dependence becomes sublinear as described by Eq. (2.46). The resistivity becomes highest when \( g/k_F = 2 \) and the modulation dependence is well approximated by Eq. (2.47). When \( g/k_F > 2 \) (\( g/k_F = 2.4 \) in Fig. 9(a)), the resistivity decreases down to a very small value and depends quadratically on the modulation strength. In this case there is no gap on the Fermi surface and scattering is possible only in higher orders as has been discussed in the previous section.

The results given in Fig. 9 show that effects of nonzero thickness are weak except when \( g/k_F > 2 \). This does not mean that the form factor can be replaced by unity but is due to complicated competition between the terms \( |A|^2 \), \( |B|^2 \), and \( |A-B|^2 \) discussed in the previous section. In fact, individual values of \( A \) and \( B \) are strongly affected by the finite thickness of the 2D system because \( F(2k_F) \sim 0.35 \).

Figure 10 shows the contribution \( \tilde{C}_{\Delta G} \) of the process with \( \Delta G \) to the coefficient \( \tilde{C} \). The contribution of process with \( \Delta G = ng \) is the same as that of process with \( \Delta G = -ng \). When \( g \) is large in the case of \( g/k_F < 2 \) such as \( g/k_F = 1.2 \) and 1.6, the system can be approximately regarded as in a weak modulation limit and the dominant contribution becomes that of the process with \( \Delta G = \pm g \). As \( g \) is decreased, the contributions of large \( \Delta G \) become important. For \( g/k_F = 0.1 \) the maximum contribution comes from processes with \( |\Delta G| = 3g \). For \( g/k_F > 2 \), the contribution for \( |\Delta G| \geq 2g \) vanishes as has been discussed in the previous section. It is noted that normal processes with \( \Delta G = 0 \) always constitute only a small part of the total contribution to the resistivity.

B. Magnetic-field modulation

Figure 11 shows the calculated coefficients \( \tilde{C} \) as a function of the modulation strength \( k_F\hbar \omega_c/g \) for various \( g \)'s. The qualitative features of dependence on \( g \) and \( k_F\hbar \omega_c/g \) are similar to those in the case of electrostatic modulation in Fig. 9 if we replace \( V_0 \) by \( k_F\hbar \omega_c/g \). When \( g \leq 2k_F \), \( \tilde{C} \) becomes larger with increasing \( g \). In Fig. 11(a), the coefficient depends almost linearly on the modulation strength for \( g/k_F = 0.1 \), but the dependence changes to being sublinear for larger \( g \). It is noted that the result for \( g/k_F = 2 \) shows a modulation-strength dependence similar to that for \( g/k_F = 1.2 \) and 1.6 in contrast to the results of electrostatic modulation. For the case of \( g/k_F = 2.4 \), \( \tilde{C} \) is considerably reduced and depends quadratically on the modulation strength. It is confirmed that the results are given well by Eq. (2.46) for \( g/k_F = 1.2 \) and 1.6, and Eq. (2.48) for \( g/k_F = 2 \). The effect of nonzero thickness is weak also.

There are quantitative differences between the results of electrostatic and magnetic-field modulations. The result for \( g/k_F = 0.1 \) for electrostatic modulation is smaller than that for magnetic-field modulation at the same width of energy gap, i.e., \( V_0 = k_F\hbar \omega_c/g \approx k_F\hbar \omega_c/g \) and the difference becomes smaller as \( g \) is increased as long as \( g/k_F < 2 \). This is partly because for small \( g/k_F \) cancellations between dominant terms in the transition probability occur in a large part of the Fermi surface \( |k_{EF} > |g/2 \) in the case of electrostatic modulation as described in Sec. IID. With the increase of \( g \), the difference becomes small partly because cancellations and enhancements become comparable and partly because main contributions come from processes related to the singular density of states where cancellations become almost irrelevant as also mentioned in Sec. IID.

C. Electrostatic and magnetic-field modulations

In actual systems with magnetic-field modulations, intrinsic electrostatic modulations are accompanied. In this case, the phase difference \( \delta \) between electrostatic and magnetic-field modulations has to be introduced which is defined by Eq. (2.2). Typical two cases with \( \delta = 0 \) and \( \pi/2 \) are studied. It is noted that the Fermi surface is asymmetric with respect to the \( k_x \) axis for \( \delta = \pi/2 \) because the modulation strength \( |V_i| \) is asymmetric as described in Sec. IID.

Figure 12 shows the calculated coefficients \( \tilde{C} \) (upper two traces) and energy-gap width \( |V_i| \) (lower three traces) as a function of strength of magnetic-field modulation \( k_F\hbar \omega_c/g \) in the case of \( g/k_F = 1.2 \) and constant strength of electrostatic modulation \( V_0/E_F = 0.0125 \). Solid lines are the results for \( \delta = 0 \) and dotted ones those for \( \delta = \pi/2 \). In the latter case, \( |V_i| \) for \( k_y > 0 \) is different from that for \( k_y < 0 \) and the gaps of the Fermi surface for \( k_y < 0 \) close off at \( k_y\hbar \omega_c/g = V_0 \), i.e., \( k_y\hbar \omega_c/gE_F \approx 0.016 \) as shown in the figure.

The result for \( \delta = \pi/2 \) has a dip at \( k_p\hbar \omega_c/gE_F \approx 0.016 \), while that for \( \delta = 0 \) is monotonic. This dip structure appears because at this magnetic-field modulation, energy gaps for negative \( k_y \) vanish and the contribution of the dominant Umklapp processes is reduced. When the magnetic-field modulation becomes sufficiently large,
\( \overline{C} \) for \( \delta = 0 \) and \( \pi/2 \) are similar to each other because energy gaps are determined mainly by the magnetic-field modulation.

IV. DISCUSSION

It is difficult to estimate the band gap in the case of electrostatic modulations, because of the screening of the 2D electron gas. In recent experiments using magnetic-field modulation caused by Ni stripes, the small electrostatic modulation has been estimated to be \( V_0/E_F \sim 0.014 \) from amplitude of the Weiss oscillation under no influence of magnetic-field modulation, i.e., when the field is applied along the stripe direction. While the absolute value of magnetic-field modulation was estimated as a function of a magnetic field perpendicular to the stripe from phase of the Weiss oscillation for coexistence of electrostatic and magnetic-field modulations (see Fig. 1(b)). Experiments were performed for the electron density \( n = 1.9 \times 10^{11} \) cm\(^{-1} \) or \( E_F = 6.8 \) meV and period \( a = 5000 \) Å. In our parameters, \( g/k_F = 0.12, \delta = \pi/2 \), and the others are the same as before.

Figure 13 shows some examples of the experimental results corresponding to \( \overline{C(h\omega_c)} - \overline{C(0)} \). The corresponding theoretical results for \( \overline{C(h\omega_c)} \) and \( \overline{C(h\omega_c)} - \overline{C(0)} \) are also shown. A small dip around \( k_F h\omega_c/gE_F \sim 0.015 \) appearing in the theoretical result is attributed to the vanishing of \( |V_0| \) for \( k_y < 0 \) similar to that shown in Fig. 12. The theoretical result is in good agreement with the experiments although slightly overestimated.

There are some possible reasons for the quantitative discrepancy. There are ambiguities in experimental estimation of the absolute magnitude of the magnetic-field modulation because it depends heavily on the estimation of the strength of electrostatic modulation based on the Weiss oscillation. There may also be ambiguities in \( T^2 \) fitting of the resistivity. Further, the calculation gives only an upper limit of the resistivity and therefore the resistivity is likely to be overestimated.

In addition to the above there are various problems to be considered. In a typical GaAs/Al\(_x\)Ga\(_{1-x}\)As heterostructure with high mobility, the elastic mean free path is about 10 \( \mu \)m and much larger than a typical potential period of 5000 Å. The inelastic mean free path determined by electron-phonon scattering has been estimated to be about 10 \( \mu \)m at 10 K in the experiments, which is much larger than a typical potential period. However, the inelastic mean free path due to electron-electron scattering is considered to be of the order of 1 \( \mu \)m at 10 K. Because scattering between plane waves going nearly parallel to the stripe direction is important when \( g/k_F \ll 1 \), the effective mean free path perpendicular to the stripe direction can be comparable to the modulation period. In this case, the gap can be reduced due to phase breaking and the resistivity becomes smaller. Finally, the temperature broadening at \( \sim 10 \) K is already comparable to the gap and therefore the temperature expansion is likely to be inadequate. When these problems are considered, slight disagreement with the experiments is rather to be expected.

The experimental results for stronger modulation using Co stripes have been also reported,\(^4\) where the maximum modulation strength can reach \( k_F h\omega_c/gE_F \approx 0.09 \). The results show similar quadratic or more stronger dependence on the strength of magnetic-field modulation. Numerical calculations in this case are quite difficult because equi-energy line of the Fermi surface becomes almost dispersionless in the \( k_x \) direction for small \( k_F \) and many singularities appear in the integrand of Eq. (2.39) due to nesting of the Fermi surface.

V. SUMMARY

In summary, the resistivity due to electron-electron scattering in 2D electron systems with weak periodic electrostatic and/or magnetic-field modulations has been studied theoretically. It has been revealed that when \( g \lesssim 2k_F \), energy gaps opening up at the boundary of the Brillouin zone play an essential role and the modulation-strength dependence of the coefficient of quadratic temperature of the resistivity is \( \overline{C\sim |V||\ln[E_F/|V||]} \) for \( g \lesssim 2k_F \) in a weak modulation limit. Numerical calculations have been systematically performed. The results are in good agreement with recent experiments.

ACKNOWLEDGMENTS

This work is supported in part by Grant-in-Aid for COE Research from the Ministry of Education, Science, Sports, and Culture (12CE2004 “Control of Electrons by Quantum Dot Structures and Its Application to Advanced Electronics”) and the JSPS ("Research for the Future" Program JSPS-RFTF96P 00103). One of the authors (S.U.) has been supported by JSPS Research Fellowships for Young Scientists.

*present address: RIKEN, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan

Appendix A: Modulation-strength dependence

Consider the representative process in the case \( g < 2k_F \) shown in Fig. 5(a). The first line of Eq. (2.40) is written for general \( \theta_1 \) and \( \theta_3 \) as

\[
\int \frac{d\delta_1d\delta_3}{|\cos(\theta_{24}/2)\sin(\theta_{24}/2)|},
\]

where \( \theta_{24} = \theta_2 - \theta_4 \) with \( \theta_2 \) and \( \theta_4 \) being angles of \( k_2 \) and \( k_3 \), respectively. They are functions of \( \theta_1 \) and \( \theta_3 \) and determined by Eq. (2.34) with \( \Delta G = g \), which is rewritten as

\[
\cos \theta_3 + \cos \theta_4 - \cos \theta_1 - \cos \theta_2 + \frac{g}{k_F} = 0,
\]

\[
\sin \theta_3 + \sin \theta_4 - \sin \theta_1 - \sin \theta_2 = 0,
\]

and therefore \( \theta_{24} \) is given by the relation

\[
\frac{\theta_{24}}{2} = \sin \frac{\theta_1 - \theta_3}{2} + \frac{g}{k_F} \sin \frac{\theta_1 + \theta_3}{2} \sin \theta_1 - \theta_3, \quad \frac{g^2}{4k_F^2}.
\]

In the case of backward scattering, \( \theta_1 \sim \theta_c \) and
where \( \theta_\lambda \) and \( \bar{\theta}_\lambda \) are the angles of \( k_c \) and \( k_c \), respectively. The integrand of Eq. (A1) becomes

\[
\frac{2}{\sqrt{[\bar{\theta}_3 - (1-g^2/2k_F^2)\bar{\theta}_3]^2 + 4gk_\|\bar{\theta}_3/k_F^2}}
\]

(A4)

where \( \bar{\theta}_1 = \theta_1 - \theta_c \) and \( \bar{\theta}_3 = \theta_3 - \bar{\theta}_c \). When \( \bar{\theta}_1 > 0 \), there is no singularity in Eq. (A4), corresponding to the fact that when \( k_\| \) lies on the + branch, the inequality \( q' < 2k_F \) holds as shown in Fig. 8(a). When \( \bar{\theta}_1 < 0 \), on the other hand, the integrand is rewritten as

\[
\frac{2}{\sqrt{(\bar{\theta}_3 - \bar{\theta}_+)(\bar{\theta}_3 - \bar{\theta}_-)}}
\]

(A5)

with

\[
\bar{\theta}_\pm = \pm \sqrt{\frac{2}{k_F^2}} |\bar{\theta}_3| + (1 - \frac{g^2}{2k_F^2}) \bar{\theta}_1,
\]

(A6)

and Eq. (A4) diverges at \( \bar{\theta}_3 = \bar{\theta}_\pm \) where the denominator vanishes. This \( \bar{\theta}_\pm \) corresponds to the condition \( q' = 2k_F \) as shown in Fig. 14. The integral should be performed in the region \( \bar{\theta}_3 \geq \bar{\theta}_+ \) or \( \bar{\theta}_3 \leq \bar{\theta}_- \).

In both cases \( \bar{\theta}_1 > 0 \) and \( \bar{\theta}_1 < 0 \), the integration of Eq. (A4) with respect to \( \bar{\theta}_3 \) gives \( \sim 2 \ln(k_\|/g_\|)|\bar{\theta}_3|) \) and its integration with respect to \( \bar{\theta}_3 \) over the region \( |\bar{\theta}_3| \leq \Delta k/k_F \) leads to Eq. (2.46).

In the case of forward scattering, \( \theta_3 \sim \pi + \bar{\theta}_3 \) with \( \pi + \bar{\theta}_3 \) being an angle of \(-k_c\). Then the density of states is approximately written as

\[
\frac{2}{\sqrt{[\bar{\theta}_3 + (1-2k_\|^2/k_F^2)\bar{\theta}_3]^2 + 2k_\|^2\bar{\theta}_3^2/k_F^2}}
\]

(A7)

where \( \bar{\theta}_3 = \theta_3 - \pi - \bar{\theta}_c \). In this case, the density of states never diverges for nonzero \( \bar{\theta}_1 \), corresponding to the fact that \( q' > 0 \) as clearly seen in Fig. 8(b). The integration of Eq. (A7) with respect to \( \bar{\theta}_3 \) gives \( \sim 2 \ln(2k_\|^2/k_F^2)|\bar{\theta}_3|) \) and its integration over \( |\bar{\theta}_3| \leq \Delta k/k_F \) again gives Eq. (2.46).

Consider the case of \( g \approx 2k_F \) next. In this case, \( \bar{\theta}_c = \bar{\theta}_c = 0 \) and \( \Delta k = \text{order of } k_F \sqrt{|V_\|/E_F} \) as shown in Eqs. (2.18) and (2.19). Because \( k_\| = 0 \), higher order terms should be considered in the denominator of Eqs. (A4) and (A7). In the case of backward scattering, the integrand becomes

\[
\frac{2}{\sqrt{\bar{\theta}_3 + \bar{\theta}_3)^2 - 4\bar{\theta}_3^2}}
\]

(A8)

The density of states becomes singular at \( \bar{\theta}_3 = \bar{\theta}_c \) and \(-\bar{\theta}_3\) corresponding to processes shown in Fig. 15 but its integral converges. The integration over \( \bar{\theta}_3 \) gives \( \sim -4 \ln |\bar{\theta}_3| \) and its integration over \( \bar{\theta}_3 \) gives Eq. (2.47) with the use of Eq. (2.18) for electrostatic modulation and Eq. (2.48) with the use of Eq. (2.19) for magnetic-field modulation, respectively.

In the case of forward scattering, the density of states is approximately given by

\[
\frac{2}{\sqrt{\bar{\theta}_3 + \bar{\theta}_3)^2 + 2\bar{\theta}_3^2}}
\]

(A9)

The integration of (A9) with respect to \( \bar{\theta}_3 \) gives \( \sim -2 \ln |\bar{\theta}_3| \) and its integration over \( \bar{\theta}_3 \) gives again Eqs. (2.47) and (2.48).

In the above calculations the change of the velocity, i.e., the change of the dispersion relation from \( h^2k^2/2m^* \), due to gap formation has been completely neglected. This corresponds to the approximation that the periodic potential \( V_i \) is set equal to zero in the integral except in the expansion coefficient \( C_{\bar{g}}(k_i) \). If such a velocity modification is considered, the divergence of the density of states can also be cut off when \( (\Delta \mathbf{v} \cdot \mathbf{u})^2 \) in the numerator becomes small and the density of states \( |v_2\times v_4|^{-1} \) is altered as well. When \( g < 2k_F \), for most of \( \theta_1 \) and \( \theta_3 \) giving a large contribution to the integral, corresponding \( \theta_2 \) and \( \bar{\theta}_3 \) usually lie outside of gap regions as already discussed in Sec. IID and therefore the velocity modification is not important.

When \( g \approx 2k_F \), the situation changes, however. In this case the divergence in the density of states appear often when all \( \theta_i \) \((j = 1, 2, 3, 4) \) lie close to or in gap regions. The velocity modification can give rise to a large change in the density \( |v_2\times v_4|^{-1} \) and the velocity factor \( (\Delta \mathbf{v} \cdot \mathbf{u})^2 \). Further, the disappearance of a part of the Fermi surface due to the gap formation (+ branch) is likely to eliminate some contributions. Therefore, the integrand in the vicinity of the divergent density of states can be different from the above expressions obtained by assuming a completely circular Fermi surface. In spite of such complications, however, the functional dependence of \( \bar{g} \) on \( V_i \) remains qualitatively same although its absolute value can be different.

In the case of backward scattering, for example, near the divergence at \( \theta_3 \sim -\bar{\theta}_1 \) in Eq. (A8), \( k_\| \) and \( k_\| \) lie near gap regions as shown in Fig. 15(a) and therefore their velocities are modified. The actual effect is not so crucial because \( k_\| \) and \( k_\| \) lie outside of gap regions. Near the divergence at \( \theta_3 \sim \bar{\theta}_1 \), \( k_\| \) and \( k_\| \) disappear because of the gap formation as shown in Fig. 15(b). In the case of forward scattering, \( k_\| \) and \( k_\| \) lie away from energy gaps as shown in Fig. 16 and the analysis with the use of circular Fermi surface is valid.

References


6. R. A. Deuschmann, W. Wegscheider, M. Rother, M. Bichler, G. Abstreiter, C. Albrecht, and J. H.
References

1. S. Uryu and T. Ando

S. Uryu and T. Ando, Proceedings of 25th International Conference of the Physics of Semiconductors, Osaka, 2000 (Springer Verlag) to be published.


22. L. P. Gor'kov and I. E. Dyaloshinsky, Pis'ma Zh. Eksp. Teor. Fiz. 18, 686 (1973) [JETP Lett. 18, 401 (1973)].


Figure Captions

Fig. 1. A schematic illustration of a 2D system with a stripe gate consisting of a magnetic material. Depending on the direction of an external magnetic field, a magnetic-field modulation with different phases is introduced. A stripe gate introduces a small electrostatic modulation due to elastic strain.

Fig. 2. Examples of the Fermi surface for (a) $g/k_F = 0.8$, $V_0/E_F = 0.1$, and $h\omega_c/E_F = 0$ and (b) its blowup of energy-gap region at $k_0$ which is enclosed by a rectangle in (a). (c) and (d) $g/k_F = 0.1$. In (b) and (d), dotted curves indicate the spherical Fermi surface.

Fig. 3. Schematic illustration of electron-electron scattering in a 2D system. Scattering is possible for (a) $k_2 \sim -k_1$ and $k_3 \sim -k_3$ and (b) $k_3 \sim k_2$ and $k_1 \sim k_1$.

Fig. 4. Processes contributing divergence of the scattering probability. (a) $g \approx 2k_F$. The Fermi surface is approximately regarded as consisting of two parallel lines as shown in (b). (c) $g \approx 0$. The Fermi surface is approximately regarded as consisting of two parallel lines as shown in (d).

Fig. 5. Schematic illustration of examples of dominant Umklapp processes where $k_1$ is fixed in an energy-gap region at $k_0$, $(k_{3x} > g/2)$ and $(G_1, G_2, G_3, G_4) = (-g, 0, 0, 0)$. Circles are the Fermi surface and dotted ones are shifted by $g$. Perpendicular lines mean the boundaries of the first Brillouin zone at $k_2 = \pm g/2$ and small circles indicate an energy-gap region. (a) $k_2 \sim k_x$, and $k_4 \sim -k_3$. (b) $k_4 \sim -k_x$ and $k_2 \sim k_3$.

Fig. 6. Scattering process with $(G_1, G_2, G_3, G_4) = (0, -g, 0, 0)$ for $k_j$ $(j = 1, \ldots, 4)$ same as in Fig. 5(a).

Fig. 7. Scattering processes with $-g/2 < k_{3x} < g/2$. (a) $(G_1, G_2, G_3, G_4) = (-g, 0, 0, 0)$. (b) $(G_1, G_2, G_3, G_4) = (0, -g, 0, 0)$.

Fig. 8. Scattering processes with (a) $k_3 \sim k_2$ and (b) $k_3 \sim -k_x$. The scattering between $k_0$ and $k_3$ occurs on two nearly parallel lines in (a) and on a single straight line in (b), giving rise to a divergence when $k_1 = k_x$.

Fig. 9. Calculated $\tilde{C}$ as a function of the modulation strength $V_0$ in the case of electrostatic modulation for (a) $g/k_F = 0.1$, 0.4, 1.2, and 2.4 and for (b) $g/k_F = 0.4, 1.2, 1.6, and 2.0$. Solid lines are the results for nonzero thickness and dotted ones those for zero thickness. A dashed line in (b) is given.
by fitting the numerical results for $g/k_F = 2$ to Eq. (2.47).

**Fig. 10.** Contribution $\tilde{C}_{\Delta G}$ of the process with $\Delta G$ to the coefficient $\tilde{C}$ in the case of electrostatic modulation with $V_0/E_F = 0.015$ for various values of $g$. Effects of nonzero thickness are taken into account.

**Fig. 11.** Calculated $\tilde{C}$ as a function of the modulation strength $k_F\hbar\omega_c/g$ in the case of magnetic-field modulation. Solid lines are the results for nonzero thickness and dotted ones those for zero thickness. A dashed line in (b) is given by fitting the numerical results for $g/k_F = 2$ to Eq. (2.48).

**Fig. 12.** Calculated $\tilde{C}$ and energy-gap width $|V_t|$ as a function of strength of magnetic-field modulation $k_F\hbar\omega_c/g$ in the presence of constant electrostatic modulation of $V_0/E_F = 0.0125$. Solid line is the result for $\delta = 0$ and dotted one that for $\delta = \pi/2$. Effects of nonzero thickness are taken into account.

**Fig. 13.** Calculated $\tilde{C}$ as a function of strength of magnetic-field modulation $k_F\hbar\omega_c/g$ in the realistic case. Effects of nonzero thickness are taken into account.

**Fig. 14.** Scattering process with $k_3 \sim k_c$ and $k_{1x} < g/2$.

**Fig. 15.** Scattering processes with (a) $\tilde{\theta}_3 \sim -3\tilde{\theta}_1$ and (b) $\tilde{\theta}_3 \sim \tilde{\theta}_1$ in the case of $g \approx 2k_F$.

**Fig. 16.** Scattering process with $k_3 \sim -k_c$ in the case of $g \approx 2k_F$. 
Fig. 1

(a) $\delta = 0$

(b) $\delta = \pi/2$

Fig. 2

(a) $g/k_F = 0.8$

(b) $g/k_F = 0.8$

(c) $g/k_F = 0.1$

(d) $g/k_F = 0.1$
Fig. 5

Fig. 6

Fig. 7
Umklapp electron-electron scattering in modulated 2D systems

Fig. 8

Fig. 9(a)

Fig. 9(b)

Fig. 10

Fig. 11(a)
Fig. 11(b)

Fig. 12

Fig. 13