Weak-Field Hall Effect in Graphene  
Calculated within Self-Consistent Born Approximation

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The weak-field Hall conductivity is calculated in monolayer graphene within a self-consistent Born approximation. The Hall conductivity sufficiently away from the Dirac point exhibits the dependence qualitatively in agreement with the Boltzmann result, but it deviates in the region near zero energy, the width of which is singularly dependent on the scattering strength. The experimental results can be reasonably well understood by assuming that the scattering strength increases with the decrease of the electron concentration and saturates in this region.

Keywords: graphene, Hall coefficient, Hall conductivity, two-dimensional graphite, level broadening, Dirac point

§1. Introduction

The graphene consisting of a monolayer graphite sheet was recently fabricated and the conductivity modulation by gate voltage was demonstrated,\(^1\) and the integer quantum Hall effect was observed.\(^2,3\) Since then, various experimental\(^{14–15}\) and theoretical\(^{16,17}\) investigations have been performed to reveal its exotic electronic and transport properties. Actually, graphene has been a subject of theoretical study prior to the experimental realization because of the peculiar electronic structure also responsible for intriguing properties of carbon nanotubes.\(^18\) The purpose of this paper is to explore singular behavior appearing in the weak-field Hall effect.

Within the effective-mass approximation or the \(k\cdot p\) scheme, the electron motion in graphene is governed by Weyl's equation for a neutrino or the Dirac equation within the effective-mass approximation or the \(k\cdot p\) scheme, the electron motion in graphene is governed by Weyl's equation for a neutrino or the Dirac equation with vanishing rest mass.\(^{16,18–24}\) An important feature is the presence of a topological singularity at \(k = 0\). This singularity is the origin of the absence of backscattering in metallic carbon nanotubes.\(^{25,26}\) It also leads to the presence of a Landau level at \(\varepsilon = 0\), responsible for the singular diamagnetic susceptibility.\(^{19,27}\) This singularity is considered as the origin of the peculiar behavior in transport, such as the minimum conductivity,\(^28\) the half-integer quantum Hall effect,\(^29\) and the dynamical conductivity.\(^30\) A massless Dirac system can also be realized in organic conductors.\(^31\)

Recently, the weak-field Hall conductivity was calculated assuming an energy independent broadening and a crossover behavior was demonstrated between the hole- and electron-like features within the energy range of the order of the broadening in the vicinity of the Dirac point.\(^32,33\) In this paper, we shall calculate the Hall conductivity in a self-consistent Born approximation for the purpose of understanding the singular behavior at the Dirac point.

It is organized as follows: In §2, a brief review is given on the electronic states, the Boltzmann conductivity, and the self-consistent Born approximation, and then the weak-field Hall conductivity is explicitly calculated. In §3 some examples of numerical results are presented on the Hall conductivity and coefficient. In §4 the results are compared with experiments and brief comments are made on the validity of the weak-field expansion. A short summary is given in §5.

§2. Formulation

2.1 Effective-mass description

In a graphite sheet the conduction and valence bands consisting of \(\pi\) orbitals cross at \(K\) and \(K'\) points of the Brillouin zone, where the Fermi level is located.\(^{34,35}\) Electronic states of the \(\pi\)-bands near a \(K\) point are described by the \(k\cdot p\) equation:\(^{16,18–24}\)

\[
\mathcal{H}_0 \Phi(r) = \varepsilon \Phi(r),
\]

(2.1)

with

\[
\mathcal{H}_0 = \gamma (\sigma \cdot \hat{k}),
\]

(2.2)

where \(\sigma = (\sigma_x, \sigma_y)\) is the Pauli spin matrix, \(\hat{k} = (\hat{k}_x, \hat{k}_y) = -i\nabla\) is a wave-vector operator, and \(\gamma\) is a band parameter, given by \(\gamma = (\sqrt{3}/2)\alpha \gamma_0\) with \(\alpha\) being the lattice constant and \(\gamma_0\) being the nearest-neighbor hopping integral.

The wave function is written as

\[
F_{s\mathbf{k}}(r) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} s \exp(i\theta(k)) \varepsilon(k) \frac{1}{L} \exp(i\mathbf{k} \cdot \mathbf{r}) \end{array} \right) ,
\]

(2.3)

where \(L^2\) is the area of the system, \(s = +1\) and \(-1\) denote the conduction and valence bands, respectively, \(k_x = k \cos \theta(k), k_y = k \sin \theta(k),\) and \(k = |k|\). The corresponding energy is given by

\[
\varepsilon_s(k) = s \gamma |k|.
\]

(2.4)

The density of states becomes

\[
D(\varepsilon) = \frac{2\varepsilon g_\varepsilon g_s |\varepsilon|}{2\pi \gamma^2} ,
\]

(2.5)

where \(g_\varepsilon = 2\) is the valley degeneracy associated with \(K\) and \(K'\) points and \(g_s = 2\) is the spin degeneracy. The
“electron” concentration is given by
\[ n_s = \int_0^{\varepsilon_F} D(\varepsilon) d\varepsilon = \frac{g_s g_0}{4\pi\gamma^2} \text{sgn}(\varepsilon_F) \varepsilon_F^2, \] (2.6)
where \( \text{sgn}(t) \) represents the sign of \( t \), i.e.,
\[ \text{sgn}(t) = \begin{cases} +1 & (t > 0); \\ 0 & (t = 0); \\ -1 & (t < 0). \end{cases} \] (2.7)
We have \( n_s < 0 \) for \( \varepsilon_F < 0 \) (hole) and \( n_s > 0 \) for \( \varepsilon_F > 0 \) (electron).

2.2 Boltzmann transport equation

The conductivity tensor is written as
\[ \sigma_{\mu\nu} = \int \left( -\frac{\partial f}{\partial \varepsilon} \right) \sigma_{\mu\nu}(\varepsilon) d\varepsilon, \quad (\mu, \nu = x, y) \] (2.8)
where \( f(\varepsilon) \) is the Fermi distribution function. The diagonal conductivity becomes
\[ \sigma_{xx}(\varepsilon) = \frac{e^2\gamma^2}{2\hbar^2} D(\varepsilon) \frac{\tau_{\text{tr}}}{1 + \omega_c^2 \tau_{\text{tr}}^2}, \] (2.9)
where \( \tau_{\text{tr}} \) is the transport relaxation time defined as
\[ \frac{1}{\tau_{\text{tr}}(\varepsilon_{\text{sk}})} = \sum_{\nu'} \int \frac{dk'}{(2\pi)^2} \langle [U_{\nu'\nu,\text{sk}}]^2 \rangle \delta[\varepsilon_{\text{sk}}(k) - \varepsilon_{\nu'(k')}], \] (2.10)
where \( U \) represents impurity potential and \( \langle \cdots \rangle \) denotes the average over impurity configurations. The Hall conductivity becomes
\[ \sigma_{xy}(\varepsilon) = -\omega_c \tau_{\text{tr}} \sigma_{xx}(\varepsilon), \] (2.11)
where \( \omega_c \) is the cyclotron frequency given by
\[ \omega_c(\varepsilon) = \frac{eB\varepsilon^2}{\epsilon\varepsilon}, \] (2.12)
with velocity \( v = \gamma/h \) and light velocity \( c \). We have \( v \approx c/300 \).

Let us confine ourselves to the case of weak magnetic field \( |\omega_c|\tau_{\text{tr}} \ll 1 \). Because of the final-state density of states in eq. (2.10), the relaxation time is usually proportional to the inverse of the density of states, i.e.,
\[ \tau_{\text{tr}} \propto D(\varepsilon_F)^{-1} \propto |\varepsilon_F|^{-1}, \] except in the case of scatterers explicitly dependent on the Fermi energy like charged impurities. As a result, the diagonal conductivity is essentially independent of the Fermi level and the electron concentration. On the other hand, the Hall conductivity is proportional to \( \varepsilon_F^2 \) through the dependence of \( \omega_c(\varepsilon_F) \) and \( \tau_{\text{tr}}(\varepsilon_F) \). In contrast to this peculiar dependence of \( \sigma_{xx} \) and \( \sigma_{xy} \) on \( \varepsilon_F \), the Hall coefficient, defined by
\[ R_H = \frac{\sigma_{xy}}{\sigma_{xx}B}, \] (2.13)
becomes
\[ R_H = -\frac{1}{n_c e^2} \] (2.14)
at zero temperature. This is exactly the same as that in conventional semiconductors and metals.

2.3 Model scatterers

There have been some theoretical investigations on effects of various kinds of scatterers, suggesting that scattering is dominated by Coulomb scattering due to charged impurities.\(^{36,37}\) Because of the screening by 2D electrons the potential range is usually smaller than the Fermi wavelength and the effective potential is approximated by
\[ U(r) = \sum_j u_j \delta(r-r_j), \]
where \( u_j \) is its strength and \( r_j \) its position. This model approximately describes charged impurities, by assuming that \( |u_j| \) effectively decreases with the increase of the electron concentration \( n_s \) roughly in proportion to \( k_F^{-1} \times n_s^{-1/2} \), where \( k_F \) is the Fermi wave vector.

For our model scatterers, the transport relaxation time becomes
\[ \frac{1}{\tau_{\text{tr}}(\varepsilon)} = \frac{\pi}{\hbar} W |\varepsilon|, \] (2.15)
where \( W \) is dimensionless parameter representing the strength of scatterers, defined as
\[ W \equiv \frac{n_i u^2}{4\pi^2 \gamma^2}, \] (2.16)
where \( u^2 = \langle u_j^2 \rangle \) and \( n_i \) is the concentration of unit area. For charged impurities, \( W \) decreases with \( n_s \) roughly in proportion to \( n_s^{-1} \). Equation (2.15) shows that \( W \ll 1 \) except in very dirty graphene such that broadening \( \hbar/\tau(\varepsilon) \) is comparable to \( \varepsilon \). Then, the diagonal conductivity becomes
\[ \sigma_{xx}(\varepsilon) = \frac{g_s g_0 e^3 B^2 \gamma^2}{4\pi^4 \hbar^2} \frac{1}{W^2 \varepsilon^2}, \] (2.17)
and the Hall conductivity is
\[ \sigma_{xy}(\varepsilon) = -\frac{g_s g_0 e^3 B^2 \gamma^2}{4\pi^4 \hbar^2} \text{sgn}(\varepsilon) \frac{1}{W^2 \varepsilon^2}, \] (2.18)
giving the Hall coefficient (2.14). It should be noted that \( \tau_{\text{tr}} \approx 2\tau \), where \( \tau \) represents the usual scattering time corresponding to single-particle lifetime. The factor two corresponds to the absence of backscattering.\(^{25,26}\)

2.4 Hall conductivity

The derivation of the Hall conductivity in the weak-field limit can be quite complicated. In a homogeneous magnetic field \( B \), the vector potential is a linear function of the position and therefore the straightforward expansion in terms of the magnetic field suffers from the problem of divergence in infinitely large systems. We can avoid this problem by using the scheme developed by Fukuyama.\(^{38,39}\)

In the present system described by the \( k \) linear Hamiltonian (2.2), the weak-field Hall conductivity is
given by
\[
\sigma_{xy} = \frac{g_\text{g} g_\text{v} \hbar}{2\pi} \int d\epsilon \left[ f(\epsilon) \frac{\partial}{\partial \epsilon} \right]_{\epsilon=\epsilon'} \left[ I_{xy}(\epsilon' - i0, \epsilon - i0) - I_{xy}(\epsilon' + i0, \epsilon + i0) \right] + \frac{\partial f(\epsilon)}{\partial \epsilon} I_{xy}(\epsilon - i0, \epsilon + i0) \right].
\] (2.19)
Here, \( I_{xy}(\epsilon', \epsilon) \) is defined by
\[
I_{xy}(\epsilon', \epsilon) = -\frac{(\epsilon - \epsilon')^2 - i}{L^2 l_\text{ch}^2} \times \langle \text{Tr} [H_x \hat{G}(\epsilon') H_y \hat{G}(\epsilon') H_y \hat{G}(\epsilon)] \rangle
- H_y \hat{G}(\epsilon) H_x \hat{G}(\epsilon') H_y \hat{G}(\epsilon) \rangle, \tag{2.20}
\]
with magnetic length \( l = \sqrt{\hbar c/eB} \) and
\[
H_\mu = \frac{\partial H}{\partial k_\mu}, \quad (\mu = x, y) \tag{2.21}
\]
\[
\hat{G}(\epsilon) = \frac{1}{\epsilon - \hat{H}}, \tag{2.22}
\]
\[
\hat{H} = \hat{H}_0 + U(r). \tag{2.23}
\]
This \( I_{xy}(\epsilon', \epsilon) \) can be represented by the diagram shown in Fig. 1 (a).

2.5 Self-consistent Born approximation

In the following we shall use Green’s function technique which allows us to consider level-broadening effects in a self-consistent manner. We shall employ the so-called self-consistent Born approximation (SCBA). This approximation is known to work sufficiently well in spite of the fact that it is the simplest without difficulty of various problems such as divergence, etc., and has already been used for the study of various properties of graphene.\(^{28,29,41,42}\)

In the self-consistent Born approximation, we consider diagrams represented in Fig. 1 (b). Then, the averaged Green’s function becomes diagonal, i.e.,
\[
\langle G_{\alpha\alpha'}(\epsilon) \rangle = \delta_{\alpha\alpha'} G_{\alpha}(\epsilon), \tag{2.24}
\]
where \( \alpha \) is a set of quantum numbers \((s, k)\). Further, for the present model scatterers, we have
\[
G_{\alpha}(\epsilon) = G_{\alpha}^0 + G_{\alpha}^0 \Sigma(\epsilon) G_{\alpha}(\epsilon), \tag{2.25}
\]
where \( G_{\alpha}^0(\epsilon) = (\epsilon - \epsilon_{\alpha})^{-1} \) and
\[
\Sigma(\epsilon) = \frac{\hbar}{4L^2} \sum_{\alpha} G_{\alpha}(\epsilon)
= \frac{2\gamma^2 W}{\pi} \int_0^{k_c} \frac{k dk}{X(\epsilon)^2 - \gamma^2 k^2}
= WX(\epsilon) \ln \left[ -\frac{\epsilon^2}{X(\epsilon)^2} \right], \tag{2.26}
\]
with
\[
X(\epsilon) = \epsilon - \Sigma(\epsilon). \tag{2.27}
\]
In the above, we have introduced cutoff wave number \( k_c \) because the integral in this equation diverges logarithmically. This divergence arises from the fact that the density of states increases linearly with energy. In actual systems, the energy band has a finite width. By considering the band edge of the \( \pi \) band, we should choose \( \epsilon_{\pi} \equiv \gamma k_c = 3\gamma_0 \).

The self-consistency equation is written as
\[
\epsilon = X + WX \ln \left( -\frac{\epsilon^2}{X^2} \right), \tag{2.28}
\]
and the Green’s function becomes
\[
G_{\alpha}(\epsilon) = \frac{1}{X(\epsilon) - \epsilon_{\alpha}}. \tag{2.29}
\]
At the Dirac point corresponding to \( \epsilon = 0 \), the self-energy and therefore \( X \) are pure imaginary. Then, the above is exactly solved as
\[
X(0) = -\Sigma(0) = i\Gamma_0, \tag{2.30}
\]
with
\[
\Gamma_0 = \epsilon_c \exp \left( -\frac{1}{2W} \right). \tag{2.31}
\]
This shows that the broadening at the Dirac point exhibits the singular dependence on scattering parameter \( W \).

Actually, the self-consistency equation can be analytically solved as
\[
X(\epsilon + i0) = \Gamma_0 \frac{\epsilon}{\epsilon_0} \frac{1}{W \text{Lambert}(\epsilon/\epsilon_0)}, \tag{2.32}
\]
where \( W \text{Lambert}(z) \) is Lambert’s \( W \) function, defined by
\[
z = W \text{Lambert}(W \text{Lambert}(z)). \tag{2.33}
\]
and
\[
\epsilon_0 \equiv 2W \Gamma_0 = 2W \epsilon_c \exp \left( -\frac{1}{2W} \right). \tag{2.34}
\]
It is interesting to note that \( X(\epsilon)/\Gamma_0 \) is a universal function of \( \epsilon/\epsilon_0 \). This means that the energy dependence of the density of states is determined by \( \epsilon_0 \), which is much smaller than broadening \( \Gamma_0 \) in clean systems \( W \ll 1 \). In fact, the density of states is given by
\[
D(\epsilon) = \frac{g_\text{g} g_\text{v}}{L^2} \sum_{\alpha} \left( -\frac{1}{\pi} \right) \text{Im} G_{\alpha}(\epsilon + i0)
= \frac{g_\text{g} g_\text{v}}{2\pi^2 \gamma^2 W} \text{Im} X(\epsilon + i0). \tag{2.35}
\]
The energy dependence of \( D(\epsilon) \) is determined by \( \epsilon_0 \) instead of \( \Gamma_0 \) and the magnitude of \( D(\epsilon) \) is scaled by \( \Gamma_0/W \), which is much larger than \( \Gamma_0 \) in clean systems.

Within SCBA, we have to consider contributions described by Figs. 1 (c), (d), and (e) for each of the square diagrams of Fig. 1 (a). The vertex corrections for a single current operator and for products of two current operators are given by diagrams shown in Figs. 1 (f) and (g), respectively. Evaluations are tedious but straightforward as in previous works.\(^{28,29,41,42}\) The results show that only vertex-corrections to a single current operator are important and that the diagrams
shown in Figs. 1 (d) and (e) give vanishing contributions. Explicitly, we have

\[
\sigma_{xy}(\epsilon) = -\frac{g_0 g_e e^3 \gamma^2}{4 \pi^2 e h^2} \left( \frac{1}{1+W} \right)^2 \left( \frac{1}{1-W} \right)^2 \left[ \frac{1}{3} \left( \frac{1}{X^2 - X^{-2}} \right) \right] \\
-\left( \frac{1}{1-W \phi(\epsilon)} \right)^2 \left( \frac{1}{X X^*} (X^2 - X^{-2}) \right),
\]

with \( X = X(\epsilon+i0), X^* = X(\epsilon-i0), \) and

\[
\phi(\epsilon) = -\frac{X X^*}{X^2 - X^{-2}} \left[ \ln(-X^2) - \ln(-X^{-2}) \right].
\]

Here, the branch of logarithm is chosen such that its imaginary part lies in \([-\pi, \pi]\). The above can be further simplified as

\[
\sigma_{xy}(\epsilon) = \frac{g_0 g_e e^3 \gamma^2}{4 \pi^2 e h^2} \frac{1}{1+W} \left( \frac{1}{1-W} \right)^2 \frac{1}{|X|^2} \\
\times \left[ -\frac{2}{3} \frac{1}{1-W} \sin 2\theta \right.
\]

\[
+ \left( \frac{1}{1-W \phi(\epsilon)} \right)^2 \left( \frac{1}{\tan 2\theta} - \frac{2\theta}{\sin^2 2\theta} \right),
\]

and

\[
\phi(\epsilon) = \frac{2\theta}{\sin 2\theta}.
\]

where angle \( \theta \) is defined by \( X = |X| e^{i\theta}. \)

In the vicinity of \( z = 0, W_L(z) \) can be expanded as

\[
W_L(z) = z - \frac{3}{2} z^3 + \cdots,
\]

giving

\[
\frac{X(\epsilon)}{\Gamma_0} \sim \frac{\epsilon}{\epsilon_0} + i \left[ 1 + \frac{1}{2} \left( \frac{\epsilon}{\epsilon_0} \right)^2 \right] + \cdots.
\]

Then, we can approximate the behavior in the energy region \( |\epsilon|/\epsilon_0 < 1 \) as

\[
\sigma_{xy}(\epsilon) \approx -\frac{g_0 g_e e^3 \gamma^2}{4 \pi^2 e h^2} \frac{4}{3 W L_0^3} \left( \frac{1}{1-W} \right)^2 \epsilon,
\]

which shows that the Hall conductivity varies linearly as a function of energy in the vicinity of the Dirac point. Further, we have

\[
n_x \approx n_0^{\gamma^2} \frac{4 \epsilon_F}{\pi \epsilon_0},
\]

with

\[
n_0 = \frac{g_0 g_e \Gamma_0^2}{4 \gamma^2}.
\]

This shows that the electron concentration for \( \epsilon_F \sim \epsilon_0 \) is given by \( n_0^{\gamma^2} \) with constant \( \alpha \sim 0.4, \) i.e., that of the ideal graphene at \( \epsilon_F \sim \Gamma_0. \) Namely, the density of states in narrow energy range \( |\epsilon| \lesssim \Gamma_0 \) of the ideal graphene is compressed into the narrower energy region \( |\epsilon| \lesssim \epsilon_0, \) by the level repulsion from the outer states. The diagonal conductivity was previously calculated as \( 28) \)

\[
\sigma_{xx}(\epsilon) = \frac{g_0 g_e e^3}{4 \pi^2 e h} \frac{1 + \phi(\epsilon)}{(1 + W)(1 - W \phi(\epsilon))},
\]

which becomes

\[
\sigma_{xx}(0) = \frac{g_0 g_e e^3}{2 \pi^2 e h} \frac{1}{1-W^2} \approx \frac{g_0 g_e e^3}{2 \pi^2 e h},
\]

at the Dirac point because \( \phi(0) = 1. \) In the energy region \( |\epsilon|/\epsilon_0 \gg 1 \) and in the clean limit \( W \ll 1, \) the results should approach those of the Boltzmann theory. In fact, we can approximately set \( X \approx \epsilon + i\pi W \epsilon \) apart from a logarithmic term, giving \( \tan \theta \approx \text{sgn}(\epsilon)/\pi W. \) Then, we have

\[
\sin 2\theta \approx -\tan 2\theta \approx 2\pi W \text{sgn}(\epsilon), \quad \theta \approx (\pi/2) \text{sgn}(\epsilon),
\]

and \( \phi(\epsilon) \approx (2W)^{-1}, \) leading to the diagonal conductivity given by eq. (2.17).

For the Hall conductivity in the clean limit, the second term in the bracket of eq. (2.38) becomes dominant and gives the Boltzmann result (2.18), i.e., \( \sigma_{xy}(\epsilon) \propto \text{sgn}(\epsilon)e^{-2}. \) To connect to the expression in the vicinity of the Dirac point, eq. (2.42), a peak in the Hall conductivity appears around \( \epsilon \sim -\epsilon_0 \) and a dip around \( \epsilon \sim +\epsilon_0. \) Factors \( [1-W \phi(\epsilon)]^{-1/2} \approx 2 \) and \( [1-W \phi(\epsilon)]^{-1/2} \approx 4, \) appearing in eqs. (2.45) and (2.38), respectively, correspond to the vertex correction converting \( \tau \) into \( \tau_\nu = 2\tau. \) In the vicinity of the Dirac point, we have \( \phi(\epsilon) \sim 1, \) showing that the vertex corrections are not important. This represents the fact that states are strongly mixed with each other and there is no difference between forward and backward scattering. In this case, both diagonal and Hall conductivity exhibits a universal dependence on energy, i.e., the energy dependence is completely scaled by \( \epsilon/\epsilon_0. \)

§3. Numerical Results

Figure 2 shows some examples of the Hall conductivity, \( \sigma_{xy}(\epsilon)W^2, \) calculated in the self-consistent Born approximation together with the Boltzmann result. The absolute value of the Hall conductivity generally remains smaller and becomes close to the Boltzmann result only in clean graphene with sufficiently small \( W \) (typically \( W < 0.01, \) although not shown here). The origin of this reduction will be discussed in more detail in the following. The energy scale near the Dirac point, \( \epsilon_0, \) is extremely small and therefore the peak and dip in \( \sigma_{xy} \) is hardly recognizable for \( W < 0.1 \). In this scale. In graphene with large disorder \( W > 0.1, \) the characteristic linear energy dependence in the vicinity of the Dirac point becomes apparent.

Figure 3 shows expansion of the results in the vicinity of the Dirac point for narrow range of \( W \sim 0.1. \) The peak in \( \sigma_{xy} \) appears at an energy slightly closer to the origin than \( -\epsilon_0 \) indicated by the upward arrows, showing that the critical region where \( \sigma_{xy} \propto \epsilon \) is slightly narrower than \( [-\epsilon_0, +\epsilon_0]. \)

Figure 4 shows the corresponding results for the density of states and the diagonal conductivity, which are the same as calculated previously.28) The most important feature is that the density of states is significantly enhanced from that in the clean system. This is mainly due to quantum-mechanical level-repulsion effect giving rise to shift of energy levels toward the Dirac point. This
enhancement of the density of states causes a significant reduction in the effective relaxation time through the enhanced final-state density of states.

For the diagonal conductivity, these changes in the density of states and the relaxation time tend to cancel each other, although being not complete, as has been discussed above. A careful analysis leads to the conclusion that the incompleteness of the cancellation and reduction of the vertex correction, i.e., \(1 - W \phi(\epsilon)\)^{-1} < 2, give almost same amount of contribution in the reduction of \(\sigma_{xx}(\epsilon)\) from the Boltzmann result.

Except in the vicinity of the Dirac point, we can expect to have relation \(\sigma_{xy} \sim -\omega_c \tau_c \sigma_{xx}\). Because the Hall conductivity is proportional to \(\tau_c\), this reduction in the relaxation time has direct influence on the absolute value of the Hall conductivity. In fact, the prominent reduction in the absolute value of \(\sigma_{yy}\) from the Boltzmann result with increasing \(W\) shown in Fig. 2 is well understood by considering the considerable decrease in the relaxation time, combined with that of the diagonal conductivity and the vertex correction.

With the use of these results and eq. (2.13), we can calculate the Hall coefficient as a function of \(n_s\). The results are shown in Fig. 5, in which the inverse of \(R_H\) is plotted against \(n_s\). In clean systems such as \(W < 0.1\), the Hall coefficient is essentially given by \(R_H = -1/n_s\) except in the critical region \(|n_s| < n_0^0\). In dirty systems such as \(W > 0.1\), \(R_H^{-1}\) considerably deviates from \(-n_s\) and there seems to be a discrete jump in the values of \(R_H^{-1}\) extrapolated to \(n_s = 0\) from the electron and hole sides. This jump may be regarded as effective carrier concentration associated with the nonvanishing density of states at the Dirac point due to disorder.

Figure 6 shows the inverse Hall coefficient versus the electron concentration in the reduced units, i.e., \(R_H^{-1}\) and \(n_s\) measured in units of \(n_0^0\) and \(n_0^0\), respectively. The figure shows clearly that the behavior in the vicinity of the Dirac point becomes nearly universal in these units and the dip and the peak of \(R_H^{-1}\) appear at \(0.2 \leq n_s/n_0^0 \leq 0.3\). It also shows that the discrete jump in the effective carrier concentration is always present and of the order of \(n_0^0\) although its absolute value becomes extremely small in the weak disorder case.

\(4\). Discussion

According to the Boltzmann result, the diagonal conductivity is independent of the carrier concentration as long as the dependence of \(W\) on \(n_s\) is not taken into account. Experimentally, however, the conductivity increases almost linearly with \(n_s\) for sufficiently large \(n_s\), showing that the effective scattering strength in actual graphene on SiO\(_2\) substrate varies considerably with \(n_s\). Most plausible scatterers giving rise to such strong \(n_s\) dependence are likely to be charged impurities.\(^{36,37}\) This strong dependence of \(W\) on \(n_s\) is expected to disappear when \(|\epsilon| < \varepsilon_0\), i.e., \(|n_s| < n_0^0\), because of the level broadening leading to uncertainty in the effective kinetic energy of electrons. Therefore, we have

\[
W \approx \begin{cases} \frac{\alpha W}{|n_s|} & (|n_s| \gg n_0^0); \\ \frac{W_0}{|n_s|} & (|n_s| \lesssim n_0^0). \end{cases}
\] (4.1)

In the experiments reported in ref. 2, \(\alpha W\) can be estimated to be \(\alpha W \sim 6 \times 10^{10} \text{ cm}^{-2}\) through the comparison with the Boltzmann conductivity in the absence of a magnetic field for \(\gamma_0 = 5 \text{ eV}\). Further, the peak and dip of \(R_H^{-1}\) appear at \(n_s \sim \pm 5 \times 10^{11} \text{ cm}^{-2}\) corresponding to \(W \approx 0.11\) shown in Fig. 5. This value is reasonably consistent with the first of Eq. (4.1) and \(W_0 \approx 0.11\).

As is clear in Fig. 5, the dependence of \(R_H^{-1}\) on \(n_s\) varies considerably between \(W = 0.11\) and 0.10 and we can practically have \(R_H^{-1} \approx -n_s\) for \(W < 0.1\) except in the extreme vicinity of \(n_s = 0\). The reason lies in the very singular dependence of \(\Gamma_0\) and \(\varepsilon_0\) on \(W\), given by eqs. (2.31) and (2.34), respectively. With the increase of \(n_s\), scattering strength \(W\) decreases from 0.1 nearly in proportion to the inverse of \(n_s\) and the Hall coefficient becomes essentially \(-1/n_s\). This is consistent with the fact that the discrete jump at the Dirac point of \(R_H^{-1}\) extrapolated to \(n_s = 0\) is not observed in the experiments.

It should be noted that similar analysis on the Hall coefficient is quite difficult for results obtained by assuming broadening independent of energy. The reason lies in inconsistent treatment of cutoff energy \(\varepsilon_c\) or cutoff wave vector \(k_c\). In fact, such cutoff is not necessary for the diagonal conductivity and the Hall conductivity, while explicit and strong dependence on the cutoff appears in the density of states and the electron concentration, giving rise to unphysical \(R_H^{-1}\) versus \(n_s\) curves near the Dirac point. Further, the singular energy dependence characterized by energy scale \(\varepsilon_0\) given by eq. (2.34) is a direct consequence of the self-consistency between the broadening and the scattering strength.

The cyclotron frequency given by eq. (2.12) diverges at \(\varepsilon = 0\), i.e., at the Dirac point. This shows that the condition for the weak magnetic field considered here may not easily be reached as was previously demonstrated.\(^{42}\) In magnetic fields, the integration over the wave vector in the self-consistency equation (2.26) is essentially replaced by the summation over discrete Landau levels \(\varepsilon_n = \text{sgn}(n|\gamma_0|)\sqrt{n\hbar B}\) with integer \(n\) and \(\hbar B = \sqrt{2}/\hbar\).\(^{29,42}\) Because \(\\text{Im}\chi(\varepsilon) \geq \Gamma_0\), the weak-field condition is satisfied in the energy range \(|\varepsilon| < \varepsilon_0\) when \(\hbar B = \Gamma_0\). For \(W = 0.1\), the broadening is \(\Gamma_0 \sim 60 \text{ meV}\) (\(B \sim 3 \text{ T}\) for \(\hbar B = \Gamma_0\)) for \(\varepsilon_c = 9 \text{ eV}\) corresponding to \(\gamma_0 = 3 \text{ eV}\) and therefore the weak-field condition can easily be reached. It may be interesting to note that this \(\Gamma_0\) is much larger than potential fluctuations \(\sim 25 \text{ meV}\) estimated from observed electron-hole puddles based on ideal density of states.\(^{11}\)

\(5\). Summary

We have calculated the weak-field Hall conductivity proportional to applied magnetic field within the self-consistent Born approximation. The results are summarized as follows:

The strength of impurity scattering is characterized...
by dimensionless parameter $W$. The Hall conductivity sufficiently away from the Dirac point exhibits the energy dependence qualitatively in agreement with the Boltzmann conductivity. Its absolute value is smaller and approaches the Boltzmann result only in the limit of weak scattering $W \ll 1$. It takes a maximum at $\varepsilon \sim -\varepsilon_0$ and a minimum at $\varepsilon \sim +\varepsilon_0$, where $\varepsilon_0 = 2W\varepsilon_0 \exp(-1/2W)$ with $\varepsilon_0$ a half of the $\pi$-band width. Between $\varepsilon \approx \pm \varepsilon_0$, it varies almost linearly as a function of energy. The scattering parameter at the Dirac point can be determined by the electron concentrations corresponding to the minimum and maximum of $R_H$. The experimental results can be reasonably well understood by $W$ which is inversely proportional to the electron concentration and saturates in the vicinity of the Dirac point.

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References


Figure Captions

Fig. 1 (a) A diagramatic representation of the weak-field Hall conductivity. (b) The average Green function in the self-consistent Born approximation. (c), (d), and (e) Diagrams for the Hall conductivity to be included in this approximation. The contributions from (d) and (e) vanish for sufficiently large cutoff energy $\varepsilon_c$. (f) The current vertex part. (g) The vertex corrections for the product of current operators.
$H_x H_y$ and $H_x H_y$ or $H_y H_x$, giving vanishingly small contributions.

Fig. 2 (Color online) Some examples of calculated Hall conductivity as a function of the Fermi energy. The dotted lines specified as “Boltzmann” represent the Hall conductivity obtained by the Boltzmann transport equation.

Fig. 3 (Color online) Some examples of the Hall conductivity in the case of large disorder, i.e., $W$ close to 0.1. The arrows indicate $-\varepsilon_0$ with $\varepsilon_0 = 2W\varepsilon_c e^{-1/2W}$.

Fig. 4 (Color online) Some examples of the density of states (dashed lines) and the conductivity (solid lines) in the absence of a magnetic field, calculated in the self-consistent Born approximation.

Fig. 5 (Color online) Some examples of the inverse of Hall coefficient $R_H$ as a function of the electron concentration. With the decrease of $W$, $R_H^{-1}$ approaches $-n_s e\varepsilon_c$. The thin solid lines are a linear extrapolation of the results for $W = 0.11$ toward $n_s = 0$.

Fig. 6 (Color online) The inverse Hall coefficient measured in units of $n_s^0 = \Gamma_0^2/\gamma^2$ is plotted against the electron concentration measured in units of $n_s^0$. 
Fig. 5

Fig. 6