Summation of series

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If a rational function $f(z)$ has poles at z_i and its denominator is two or more degrees greater than the numerator, then we have due to the residue theorem

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_i \mathrm{Res}\left(f(z)\frac{\pi}{\tan \pi z}; z_i\right), \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_i \mathrm{Res}\left(f(z)\frac{\pi}{\sin \pi z}; z_i\right),$$
$$\sum_{n=-\infty}^{\infty} f(n)e^{inx} = -\sum_i \mathrm{Res}\left(f(z)\frac{2\pi i e^{izx}}{e^{2\pi i z}-1}; z_i\right), \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n)e^{inx} = -\sum_i \mathrm{Res}\left(f(z)\frac{\pi e^{izx}}{\sin \pi z}; z_i\right). \tag{1}$$

If $f(z)$ has poles at $z = n$(integers), these integers should be subtracted on the left hand sides.

Show

$$\sum_{n=1}^{\infty} \frac{2}{n^{2m}} = \frac{B_m (2\pi)^{2m}}{(2m)!}, \quad \sum_{n=-\infty}^{\infty} \frac{1}{(3n-1)^2} = \frac{4\pi^2}{27}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth \pi a - \frac{1}{2a^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a \sinh \pi a} - \frac{1}{2a^2},$$
$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh(\pi - x)a}{\sinh \pi a} - \frac{1}{2a^2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \sin nx}{n^2+a^2} = \frac{\pi}{2a}\frac{\sinh ax}{\sinh \pi a}. \tag{2}$$

Here, B_m is the Bernoulli number.