Elliptic integrals

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LENGTH OF SINE CURVE

Consider a sine curve given by

$$y = b \sin \frac{x}{a}. \tag{1}$$

Its line element reads

$$ds^2 = \frac{a^2+b^2}{a^2}\left(1 - k^2 \sin^2 \frac{x}{a}\right) dx^2, \quad k^2 = \frac{b^2}{a^2+b^2}. \tag{2}$$

Therefore, the length of the sine curve over a quarter period can be calculated as

$$\frac{\sqrt{a^2+b^2}}{a}\int_0^{\frac{\pi}{2}a}\sqrt{1-k^2\sin^2\frac{x}{a}}dx = \sqrt{a^2+b^2}\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}d\varphi = \sqrt{a^2+b^2}E(k) \tag{3}$$

where $E(k)$ is the complete elliptic integral of the second kind.

MAGNETOSTATIC POTENTIAL BY A CHARGED CIRCLE

Let us consider the magnetostatic potential induced by charge uniformly distributed with unit line density on a circle with radius R. We consider the potential at a point $\mathrm{P}(a, b, c)$. The point on the circle can be written as $\mathrm{Q}(R\cos\theta, R\sin\theta, 0)$. Then, we have

$$\mathrm{PQ}^2 = \left(\sqrt{a^2+b^2}+R\right)^2 + c^2 - 4R\sqrt{a^2+b^2}\cos^2\frac{\theta-\alpha}{2} \tag{4}$$

with $\tan\alpha = \frac{b}{a}$. Therefore, we can calculate the potential at P as

$$\begin{aligned} U = \frac{1}{4\pi\varepsilon}\int_0^{2\pi}\frac{Rd\theta}{\mathrm{PQ}} &= \frac{R}{2\pi\varepsilon}\int_0^{\pi}\frac{d\theta}{\sqrt{\left(\sqrt{a^2+b^2}+R\right)^2+c^2-4R\sqrt{a^2+b^2}\cos^2\frac{\theta}{2}}} \\ &= \frac{R}{\pi\varepsilon}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{\left(\sqrt{a^2+b^2}+R\right)^2+c^2-4R\sqrt{a^2+b^2}\sin^2\varphi}} \\ = \frac{R}{\pi\varepsilon}\frac{1}{\sqrt{\left(\sqrt{a^2+b^2}+R\right)^2+c^2}}&\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{k}{2\pi\varepsilon}\sqrt{\frac{R}{\sqrt{a^2+b^2}}}K(k) \end{aligned} \tag{5}$$

where

$$k^2 = \frac{4R\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}+R\right)^2+c^2} \tag{6}$$

and $K(k)$ is the complete elliptic integral of the first kind. In the limit of $a, b \to 0$, we have $U = \frac{R}{2\varepsilon\sqrt{R^2+c^2}}$.

ANHARMONIC OSCILLATION

Here, let us consider the oscillation of a particle confined by anharmonic potentials. The law of energy conservation reads

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x). \tag{7}$$

We can factorize as

$$\frac{2}{m}(E - U(x)) = (x - a_1)(a_2 - x)V(x). \tag{8}$$

We assume that the oscillation occurs between a_1 and a_2 ($a_1 < a_2$). By integrating Eq.(7), we have

$$t = \int \frac{dx}{\sqrt{(x - a_1)(a_2 - x)V(x)}}. \tag{9}$$

By setting $x = \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_1 - a_2)\cos\varphi$, we finally obtain the following expression of t:

$$t = \int \frac{d\varphi}{\sqrt{V}}. \tag{10}$$

Quartic potential. As an example, let us consider a quartic potential of the form

$$U(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 \tag{11}$$

with $\alpha, \beta > 0$, which can be rewritten as

$$\frac{2}{m}(E - U(x)) = -\frac{\beta}{2m}(x^2 - a^2)(b^2 + x^2) \tag{12}$$

where $E > 0$ and

$$a^2 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta E}}{\beta}, \quad b^2 = a^2 + \frac{2\alpha}{\beta}. \tag{13}$$

The period of the oscillation is thus calculated as

$$T = 2\sqrt{\frac{2m}{\beta}}\int_0^\pi \frac{d\varphi}{\sqrt{b^2 + a^2\cos^2\varphi}} = 4\sqrt{\frac{2m}{\beta(a^2 + b^2)}}K(k) = 4\sqrt{\frac{m}{\sqrt{\alpha^2 + 4\beta E}}}K(k) \tag{14}$$

where

$$k^2 = \frac{a^2}{a^2 + b^2} = \frac{\beta a^2}{2(\alpha + \beta a^2)}. \tag{15}$$

In the limit of $\beta \to 0$, we have $T = 2\pi\sqrt{\frac{m}{\alpha}}$.

Cubic potential. Next, let us consider a cubic potential of the form

$$U(x) = \frac{\alpha}{2}x^2 - \frac{\gamma}{2}x^3 \tag{16}$$

with $\alpha, \gamma > 0$, which can be factorized as

$$\frac{2}{m}(E - U(x)) = \frac{\gamma}{m}(x - a_1)(a_2 - x)(a_3 - x), \quad a_1 < a_2 < a_3 \tag{17}$$

where $E > 0$, and thus $V(x) = \frac{\gamma}{m}(a_3 - x)$. The period of the oscillation is given by

$$T = 2\sqrt{\frac{m}{\gamma}}\int_0^\pi \frac{d\varphi}{\sqrt{a_3 - \frac{1}{2}(a_1 + a_2) + \frac{1}{2}(a_2 - a_1)\cos\varphi}} = 4\sqrt{\frac{m}{\gamma(a_3 - a_1)}}K(k) \tag{18}$$

where $k^2 = \frac{a_2 - a_1}{a_3 - a_1}$. In the limit of $\gamma \to 0$, we have $a_1 = -a_2$ and $\gamma a_3 = \alpha$, and hence $T = 2\pi\sqrt{\frac{m}{\alpha}}$.

ROTATION OF A RIGID BODY

Consider a free rotation of a rigid body described by the Euler's equation of motion in a rotating frame: $\frac{d}{dt}\mathbf{L}+\omega\times\mathbf{L}=0$ where

$$\omega=\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix},\quad \mathbf{L}=\begin{pmatrix}A\omega_1\\ B\omega_2\\ C\omega_3\end{pmatrix}. \tag{19}$$

Thus, we have

$$A\frac{d}{dt}\omega_1=(B-C)\omega_2\omega_3,\quad B\frac{d}{dt}\omega_2=(C-A)\omega_3\omega_1,\quad C\frac{d}{dt}\omega_3=(A-B)\omega_1\omega_2. \tag{20}$$

For $C>B>A$, we consider the solution of the form:

$$\omega_1=\alpha\mathrm{cn}\lambda t,\quad \omega_2=\beta\mathrm{sn}\lambda t,\quad \omega_3=\gamma\mathrm{dn}\lambda t. \tag{21}$$

Note $\omega_3>0$. Inserting these ansatz into the equation of motion, we have

$$\frac{\alpha\beta\gamma}{\lambda}=\frac{A\alpha^2}{C-B}=\frac{B\beta^2}{C-A}=\frac{k^2C\gamma^2}{B-A}. \tag{22}$$

The energy conservation law reads $2E=A\alpha^2+C\gamma^2$ with the energy E. The magnitude of angular momentum is also conserved: $L^2=A^2\alpha^2+C^2\gamma^2$. With these conditions, we finally obtain

$$\alpha^2=\frac{2EC-L^2}{A(C-A)},\quad \beta^2=\frac{2EC-L^2}{B(C-B)},\quad \gamma^2=\frac{-2EA+L^2}{C(C-A)},\quad k^2=\frac{B-A}{C-B}\frac{2EC-L^2}{-2EA+L^2},\quad \lambda^2=\frac{C-B}{ABC}(-2EA+L^2) \tag{23}$$