Some formulae

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We here show some useful formulae.

DIFFERENTIAL OPERATORS ON CURVILINEAR COORDINATES

New, consider an orthogonal curvilinear coordinate described as

$$\mathbf{A} = A^i \mathbf{e}_i, \quad \mathbf{e}_i = \frac{1}{h^i}\frac{\partial \mathbf{x}}{\partial q^i}, \quad h_i = \left|\frac{\partial \mathbf{x}}{\partial q^i}\right|. \tag{1}$$

Then, we have

$$\mathrm{rot}\mathbf{A} = \frac{1}{h_2 h_3}\left\{\frac{\partial(h_3 A_3)}{\partial q^2} - \frac{\partial(h_2 A_2)}{\partial q^3}\right\}\mathbf{e}_1 + \frac{1}{h_3 h_1}\left\{\frac{\partial(h_1 A_1)}{\partial q^3} - \frac{\partial(h_3 A_3)}{\partial q^1}\right\}\mathbf{e}_2 + \frac{1}{h_1 h_2}\left\{\frac{\partial(h_2 A_2)}{\partial q^1} - \frac{\partial(h_1 A_1)}{\partial q^2}\right\}\mathbf{e}_3. \tag{2}$$

In general curvilinear coordinate systems, the Laplace operator reads

$$\Delta = \sum_{a=1}^{D}\left(\frac{\partial}{\partial x^a}\right)^2 = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^\nu}\sqrt{g}g^{\nu\lambda}\frac{\partial}{\partial \xi^\lambda}, \tag{3}$$

where $x^a \to \xi^\nu, a, \nu = 1, 2, ..., D$, x^a and ξ^ν denote the Euclidean and a general curvilinear coordinates, respectively, and

$$g^{\mu\nu} = \frac{\partial \xi^\mu}{\partial x^a}\frac{\partial \xi^\nu}{\partial x^a}, \quad g_{\mu\nu} = \frac{\partial x^a}{\partial \xi^\mu}\frac{\partial x^a}{\partial \xi^\nu}, \quad g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda \tag{4}$$

$$g = \det(g_{\mu\nu}), \quad dx^a = \frac{\partial x^a}{\partial \xi^\mu}d\xi^\mu, \quad ds^2 = dx^a dx^a = g_{\mu\nu}d\xi^\mu d\xi^\nu. \tag{5}$$

GAUSS-LEGENDRE MULTIPLICATION FORMULA

The gamma function $\Gamma(z)$ satisfies the Gauss-Legendre multiplication formula:

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{(n-1)/2}}\prod_{k=0}^{n-1}\Gamma(z + k/n). \tag{6}$$

BESSEL FUNCTIONS

Several types of equations can be reduced to the Bessel equations. The solution of the equation

$$x^2 y'' + (1-2a)xy' + \left[(bcx^c)^2 + a^2 - \alpha^2 c^2\right]y = 0 \tag{7}$$

is given by $y = x^a J_\alpha(bx^c)$. As a corollary, $y = e^{ax}J_\alpha(be^{cx})$ is the solution of

$$y'' - 2ay' + \left[(bce^{cx})^2 + a^2 - \alpha^2 c^2\right]y = 0. \tag{8}$$

Now, consider $y = J_\alpha(f(x))$. Then, y satisfies $(' = \frac{d}{dx})$

$$y'' + \frac{(f')^2 - ff''}{ff'}y' + \left(\frac{f'}{f}\right)^2(f^2 - \alpha^2)y = 0. \tag{9}$$

By substituting $f(x) = be^{cx}$, we find that $y = J_\alpha(be^{cx})$ is the solution of the equation

$$y'' + c^2 \left[b^2 e^{2cx} - \alpha^2\right] y = 0. \tag{10}$$

Also, $y = g(x)J_\alpha(bx)$ satisfies

$$x^2 y'' + \left(x - 2x^2 \frac{g'}{g}\right) y' + \left[\left(b^2 + \frac{2(g')^2 - gg''}{g^2}\right) x^2 - \frac{g'}{g} x - \alpha^2\right] y = 0. \tag{11}$$

By setting $g(x) = e^{ax}$, we see that $y = e^{ax} J_\alpha(bx)$ is the solution of the equation

$$x^2 y'' + \left(x - 2ax^2\right) y' + \left[\left(a^2 + b^2\right) x^2 - ax - \alpha^2\right] y = 0. \tag{12}$$

One can also show that $y = [h(x)]^\beta J_\alpha(h(x))$ satisfies

$$y'' + \left[(1 - 2\beta)\frac{h'}{h} - \frac{h''}{h'}\right] y' + \left(\frac{h'}{h}\right)^2 (h^2 + \beta^2 - \alpha^2) y = 0. \tag{13}$$

Solve the following equations:

$$\begin{gathered} xy'' + y' + y = 0,\ xy'' - y' + xy = 0,\ y'' + xy = 0,\ y'' + e^{2x} y = 0, \\ xy'' + (x+1)y' + (x + 1/2)y = 0,\ y'' + \tan x y' + \cos^2 x y = 0,\ y'' + \lambda^2 x^{p-2} y = 0. \end{gathered} \tag{14}$$

The solution of the following equation

$$\frac{d^{2n} y}{dx^{2n}} = (-1)^n \lambda^{2n} x^{-n} y. \tag{15}$$

is given by $y = x^{n/2} J_n(2\lambda x^{1/2})$.

Asymptotic expansion for large order is given by

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu,\ \nu \to \infty. \tag{16}$$

Let us consider integral of triple products of Bessel functions. Let $A = s(s-a)(s-b)(s-c)$ and $s = (a+b+c)/2$. If $\mathrm{Re}\nu > -1/2$, then we have

$$\int_0^\infty J_\nu(at) J_\nu(bt) J_\nu(ct) t^{1-\nu} dt = \begin{cases} \frac{2^{\nu-1} A^{\nu-1/2}}{\sqrt{\pi}(abc)^\nu \Gamma(\nu+1/2)},\ A > 0 \\ 0,\ A \le 0 \end{cases}. \tag{17}$$

Consider the Taylor's expansion of $(z+h)^{-\nu/2} J_\nu(\sqrt{z+h})$ with respect to h:

$$(z+h)^{-\nu/2} J_\nu(\sqrt{z+h}) = \sum_{n=0}^\infty \frac{h^m}{m!} \frac{d^m}{dz^m} z^{-\nu/2} J_\nu(\sqrt{z}) = \sum_{n=0}^\infty \frac{(-h/2)^m}{m!} z^{-(m+\nu)/2} J_{\nu+m}(\sqrt{z}). \tag{18}$$

Similarly, when $|h| < |z|$, we have the expansion

$$(z+h)^{\nu/2} J_\nu(\sqrt{z+h}) = \sum_{n=0}^\infty \frac{h^m}{m!} \frac{d^m}{dz^m} z^{\nu/2} J_\nu(\sqrt{z}) = \sum_{n=0}^\infty \frac{(h/2)^m}{m!} z^{(\nu-m)/2} J_{\nu-m}(\sqrt{z}) \tag{19}$$

Setting $\nu = -1/2$ and $\nu = 1/2$ and making some variable replacements in the above equations, we obtain

$$\sqrt{\frac{2}{\pi z}} \cos\sqrt{z^2 - 2zt} = \sum_{n=0}^\infty \frac{t^m}{m!} J_{m-1/2}(z),\ \sqrt{\frac{2}{\pi z}} \sin\sqrt{z^2 + 2zt} = \sum_{n=0}^\infty \frac{t^m}{m!} J_{1/2-m}(z). \tag{20}$$

The latter is valid as long as $2\,|t| < |z|$. Also, with the substitution $z \to z^2$ and $h \to kz^2$, we have

$$J_\nu(\sqrt{1+k}z) = (1+k)^{\nu/2} \sum_{n=0}^\infty \frac{(-kz/2)^m}{m!} J_{\nu+m}(z). \tag{21}$$

By putting $\sqrt{1+k} = \lambda$, we have

$$J_\nu(\lambda z) = \lambda^\nu \sum_{n=0}^{\infty} \frac{(1-\lambda^2)^m (z/2)^m}{m!} J_{\nu+m}(z). \tag{22}$$

This is called the multiplicaton theorem.

Now, define the functions $O_n(t), n = 0, 1, ...$ as

$$O_0(t) = \frac{1}{t}, \quad O_1(t) = -O_0'(t), \quad O_{n+1}(t) = O_{n-1}(t) - 2O_n'(t). \tag{23}$$

For Re$t > 0$, $O_n(t)$ can be expressed as

$$O_n(t) = \frac{1}{2} \int_0^\infty e^{-tu} \left\{ \left(u + \sqrt{u^2+1}\right)^n + \left(u - \sqrt{u^2+1}\right)^n \right\} du. \tag{24}$$

The generating function is given by (check this by calculating the r.h.s directly or showing that the r.h.s is solely a function of $t - z$)

$$\frac{1}{t-z} = J_0(z)O_0(t) + 2\sum_{n=1}^{\infty} J_n(z)O_n(t). \tag{25}$$

If $f(z)$ is holomorphic in $|z| < R$, then we have the expansion

$$f(z) = c_0 J_0(z) + 2\sum_{n=1}^{\infty} c_n J_n(z),\ c_n = \frac{1}{2\pi i} \int_{|t|=R} f(t)O_n(t)dt. \tag{26}$$

In particular, we have

$$2\pi i J_m(z) = J_0(z) \int_{|t|=R} J_m(t)O_0(t)dt + 2\sum_{n=1}^{\infty} J_n(z) \int_{|t|=R} J_m(t)O_n(t)dt. \tag{27}$$

BLOCK MATRICES

A block matrix can be decomposed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}. \tag{28}$$

Thus, we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BD^{-1}CD). \tag{29}$$

Inverse of a block matrix can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}. \tag{30}$$

In particular, we have

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}. \tag{31}$$