The Hankel transform

Takehito Yokoyama
Department of Physics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan
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The νth order Hankel transform of $f(r)$ is defined as

$$F_\nu(k) = \mathcal{H}_\nu\left[f(r)\right] = \int_0^\infty r f(r) J_\nu(kr) dr. \tag{1}$$

The inverse Hankel transform is given by

$$f(r) = \int_0^\infty k F_\nu(k) J_\nu(kr) dk. \tag{2}$$

The inversion formula is valid for $\nu > -1/2$.

Properties of Hankel transforms.

Hankel transforms have the following properties. Define Δ_ν as

$$\Delta_\nu = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\nu^2}{r^2}. \tag{3}$$

Then, if $\lim_{r\to\infty} f(r) = 0$, we have

$$\mathcal{H}_\nu\left[\Delta_\nu f(r)\right] = -k^2 F_\nu(k). \tag{4}$$

$$\mathcal{H}_\nu\left[f'(r)\right] = \frac{k}{2\nu}\left[(\nu+1)F_{\nu-1}(k) - (\nu-1)F_{\nu+1}(k)\right],\ \mathcal{H}_\nu\left[\frac{1}{r}f(r)\right] = \frac{k}{2\nu}\left[F_{\nu-1}(k) + F_{\nu+1}(k)\right]. \tag{5}$$

$$\mathcal{H}_\nu\left[\frac{1}{r^2}f(r)\right] = \frac{k^2}{4\nu(\nu-1)(\nu+1)}\left[(\nu+1)F_{\nu-2}(k) + 2\nu F_\nu(k) + (\nu-1)F_{\nu+2}(k)\right]. \tag{6}$$

$$\mathcal{H}_\nu\left[r^{\nu-1}\frac{d}{dr}r^{1-\nu}f(r)\right] = -kF_{\nu-1}(k),\ \mathcal{H}_\nu\left[r^{-\nu-1}\frac{d}{dr}r^{\nu+1}f(r)\right] = kF_{\nu+1}(k). \tag{7}$$

Let $G_\nu(k) = \mathcal{H}_\nu\left[g(r)\right]$. Then, we have the Parseval's theorem of the form:

$$\int_0^\infty r f(r) g(r) dr = \int_0^\infty k F_\nu(k) G_\nu(k) dk. \tag{8}$$

Examples of Hankel transforms.

$$\mathcal{H}_0\left[e^{-ar^2}\right] = \frac{1}{2a}e^{-k^2/4a},\ a > 0. \tag{9}$$

$$\mathcal{H}_0\left[\delta(r-a)\right] = aJ_\nu(ka),\ \mathcal{H}_0\left[aJ_\nu(ar)\right] = \delta(k-a),\ a > 0. \tag{10}$$

$$\mathcal{H}_\nu\left[r^{\nu-1}e^{-ar}\right] = \int_0^\infty r^\nu e^{-ar} J_\nu(kr) dr = \frac{1}{k^{\nu+1}}\mathcal{L}\left[x^\nu J_\nu(x), p = a/k\right]. \tag{11}$$

Here, $\mathcal{L}$ is the Laplace transform. Performing the Laplace transform of $x^\nu J_\nu(x)$, we have

$$\mathcal{L}\left[x^\nu J_\nu(x), p\right] = \frac{2^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi}(p^2+1)^{\nu+1/2}} \tag{12}$$

and, therefore,

$$\mathcal{H}_\nu\left[r^{\nu-1}e^{-ar}\right] = \frac{(2k)^\nu \Gamma(\nu + 1/2)}{\sqrt{\pi}(a^2+k^2)^{\nu+1/2}}. \tag{13}$$

Applications of Hankel transforms.

1. Consider the heat conduction problem in a steady state:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) u = 0,\ z > 0 \tag{14}$$

with the boundary condition,

$$-\kappa \frac{\partial u}{\partial z} = q\theta(a-r),\ z = 0. \tag{15}$$

With the Hankel transform ($u \to U$), the problem is reduced to

$$\left(-k^2 + \frac{\partial^2}{\partial z^2}\right) U = 0, \tag{16}$$

$$-\kappa \frac{\partial U}{\partial z} = qaJ_1(ka)/k,\ z = 0. \tag{17}$$

By solving the above equation and performing the inverse transform, we finally obtain the solultion of the form

$$u = qa/\kappa \int_0^\infty e^{-kz} J_1(ka) J_0(kr)/k dk. \tag{18}$$

2. Consider a similar problem:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) u(r,z) = 0,\ z > 0 \tag{19}$$

with the boundary condition

$$u(r,0) = f(r),\ z = 0. \tag{20}$$

With the Hankel transform, we obtain

$$U(k,z) = e^{-kz} \int_0^\infty r f(r) J_0(kr) dr. \tag{21}$$

Hence, we have

$$u(r,z) = \int_0^\infty k e^{-kz} J_0(kr) dk \int_0^\infty s f(s) J_0(ks) ds. \tag{22}$$

3. Consider the electrostatic potential between two grounded horizontal plates at $z = \pm l$ in the presence of a point charge q at $r = z = 0$. The potential u can be written as

$$u = \varphi + q(r^2 + z^2)^{-1/2} \tag{23}$$

where φ obeys the Laplace equation. The boundary conditions read

$$\varphi(r, \pm l) + q(r^2 + l^2)^{-1/2} = 0. \tag{24}$$

Taking the Hankel transform, these are reduced to

$$\left(-k^2+\frac{\partial^2}{\partial z^2}\right)\Phi=0,\ \Phi(k,\pm l)=-\frac{qe^{-kl}}{k}. \tag{25}$$

Thus, we have

$$\Phi(k,z)=-\frac{qe^{-kl}}{k}\frac{\cosh(kz)}{\cosh(kl)} \tag{26}$$

and hence

$$\varphi=q(r^2+z^2)^{-1/2}-q\int_0^\infty e^{-kl}\frac{\cosh(kz)}{\cosh(kl)}J_0(kr)dk. \tag{27}$$

Finite Hankel transforms.

Let $\beta_{\nu,n}$ the solution of $hJ_\nu(x)+xJ'_\nu(x)=0$. Define the finite Hankel transform as

$$F_\nu(\beta_{\nu,n})=\int_0^1 rf(r)J_\nu(\beta_{\nu,n}r)dr. \tag{28}$$

Then, the inverse transform is given by

$$f(r)=2\sum_{n=1}^\infty\frac{(\beta_{\nu,n})^2F_\nu(\beta_{\nu,n})}{h^2+(\beta_{\nu,n})^2-\nu^2}\frac{J_\nu(\beta_{\nu,n}r)}{(J_\nu(\beta_{\nu,n}))^2}. \tag{29}$$

This series is called the Dini expansion of $f(r)$. Note the orthogonality

$$\int_0^1 rJ_\nu(\beta_{\nu,m}r)J_\nu(\beta_{\nu,n}r)dr=\frac{1}{2(\beta_{\nu,n})^2}\left[\left(\beta_{\nu,n}J'_\nu(\beta_{\nu,n})\right)^2+\left((\beta_{\nu,n})^2-\nu^2\right)\left(J_\nu(\beta_{\nu,n})\right)^2\right]\delta_{nm}. \tag{30}$$

As an example, consider the heat conduction problem

$$\kappa\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)u=\frac{\partial}{\partial t}u,\ 0\le r<1. \tag{31}$$

with the initial condition

$$u(r,0)=1, 0\le r\le 1 \tag{32}$$

and the boundary condition

$$\frac{\partial u}{\partial r}=-hu,\ r=1 \tag{33}$$

where h is a positive constant. By the transformation of the differential equation, we have

$$\frac{dU}{dt}=-\kappa(\beta_{0,n})^2U \tag{34}$$

and thus

$$U=\frac{J_1(\beta_{0,n})}{\beta_{0,n}}e^{-\kappa(\beta_{0,n})^2t} \tag{35}$$

Using the inversion formula, we have

$$u=2\sum_{n=1}^\infty e^{-\kappa(\beta_{0,n})^2t}\frac{\beta_{0,n}J_1(\beta_{0,n})}{h^2+(\beta_{0,n})^2}\frac{J_0(\beta_{0,n}r)}{(J_0(\beta_{0,n}))^2}. \tag{36}$$

The Weber transform.

Define

$$Z_\nu(k,r) = J_\nu(kr)Y_\nu(k) - Y_\nu(kr)J_\nu(k). \tag{37}$$

Then, the Weber transform is given by

$$F_\nu(k) = \int_1^\infty r f(r) Z_\nu(k,r) dr,\ f(r) = \int_0^\infty k F_\nu(k) \frac{Z_\nu(k,r)}{(J_\nu(k))^2 + (Y_\nu(k))^2} dk. \tag{38}$$

If f is given by

$$f(x) = g''(x) + \frac{1}{x} g'(x) - \frac{\nu^2}{x^2} g(x), \tag{39}$$

then, we have

$$F_\nu(k) = -k^2 G_\nu(k) - \frac{2}{\pi} g(1). \tag{40}$$