Derivation of Laplacian in 3D polar coordinate

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(Dated: November 14, 2016)

Consider orthonormal basis in 3D polar coordinate:

$$\alpha = dr, \quad \beta = r d\theta, \quad \gamma = r \sin\theta d\varphi. \tag{1}$$

The Hodge star operator acts on these 1-forms as

$$*\alpha = \beta \wedge \gamma, \quad *\beta = \gamma \wedge \alpha, \quad *\gamma = \beta \wedge \alpha, \quad *1 = \alpha \wedge \beta \wedge \gamma \tag{2}$$

leading to

$$*dr = r^2 \sin\theta d\theta \wedge d\varphi, \quad *d\theta = \sin\theta d\varphi \wedge dr, \quad *d\varphi = \frac{1}{\sin\theta} dr \wedge d\theta, \quad *dr \wedge d\theta \wedge d\varphi = \frac{1}{r^2 \sin\theta}. \tag{3}$$

For a function f, we have

$$*df = r^2 \sin\theta \frac{\partial}{\partial r} f d\theta \wedge d\varphi + \sin\theta \frac{\partial}{\partial\theta} f d\varphi \wedge dr + \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi} f dr \wedge d\theta \tag{4}$$

and

$$*d*df = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} f + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} f + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2} f. \tag{5}$$

The Laplacian is defined as $\Delta = \delta d + d\delta$ where δ is the codifferential: $\delta = - * d *$. Since $d * f = 0$, we finally obtain

$$\Delta = -\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} - \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\varphi^2}. \tag{6}$$