Geometry, conformal invariance, and complex analysis

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HOMOLOGY, COHOMOLOGY, STOKES' THEOREM, POINCARE'S LEMMA, DE RHAM'S THEOREM, AND DOLBEAULT'S LEMMA

Illustrate some examples with the boundary operator ∂. The cycle and boundary cycle are defined by $\partial C_r = 0$ and $C_r = \partial C_{r+1}$, respectively (r is the dimension). If $C_r - C_r' = \partial C_{r+1}$, these are homologue, which gives homology group, $H = Z/B$ (cycle/boundary cycle). From the definition, we have $\partial^2 = 0$.

As for cohomology, similarly we define $d\omega^r = 0$ and $\omega^{r+1} = d\omega^r$ as closed and exact forms (cocycle and boundary cocycle), respectively. When $\omega^{r+1} - \omega'^{r+1} = d\omega^r$, these are cohomologue, which leads to cohomology group, $H = Z/B$ (closed/exact).

Show some examples with the exterior differentiation operator d. Let us first consider 0-form in 2D $\varphi = f(x, y)$. Then,

$$d\varphi = \partial_x f(x, y)dx + \partial_y f(x, y)dy. \tag{1}$$

Introduce "wedge" $\wedge$. Note that $dx \wedge dy = -dy \wedge dx$ etc. From the definition, we have

$$d^2\varphi = (\partial_x \partial_y f(x, y) - \partial_y \partial_x f(x, y))dx \wedge dy = 0. \tag{2}$$

Generally, we find $d^2 = 0$.

Now, let us consider 1-form $\varphi = f_x dx + f_y dy$. Then,

$$d\varphi = (\partial_y f_x - \partial_x f_y)dx \wedge dy = (\text{rot}\mathbf{f})_z dx \wedge dy. \tag{3}$$

Thus, rotation free vectors give closed forms. For instance, $\varphi = \frac{-ydx+xdy}{x^2+y^2}$ is closed. Let us consider 2-form in 3D

$$\varphi = f_x dy \wedge dz + f_y dz \wedge dx + f_z dx \wedge dy, \tag{4}$$

and then we have

$$d\varphi = (\partial_x f_x + \partial_y f_y + \partial_z f_z)dx \wedge dy \wedge dz = (\text{div}\mathbf{f})dx \wedge dy \wedge dz. \tag{5}$$

Thus, divergenceless vectors give closed forms.

Generalized Stokes' theorem is represented as

$$\int_{C_{r+1}} d\omega^r = \int_{\partial C_{r+1}} \omega^r \tag{6}$$

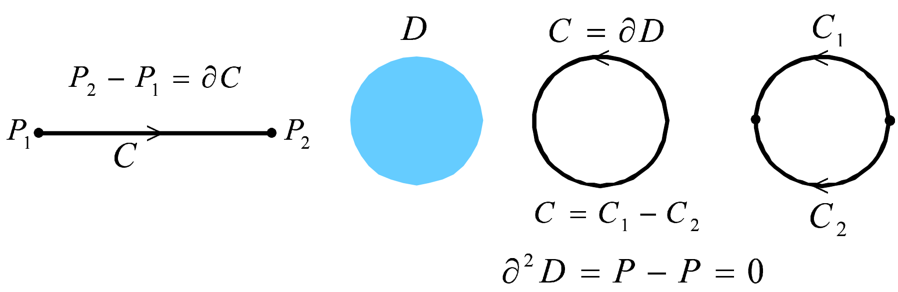

FIG. 1: Some examples.

or simply $(C_{r+1}, d\omega^r) = (\partial C_{r+1}, \omega^r)$. Show examples for $r = 0, 1, 2$. We see the similarity between ∂ and d.

The Stokes' theorem can be extended to complex variables, which immediately provides the complex form of Green's theorem:

$$\int_{\partial C_2} f(z,\bar{z})dz = \int_{C_2} \frac{\partial}{\partial \bar{z}} f(z,\bar{z}) d\bar{z} \wedge dz. \tag{7}$$

Note $d\bar{z} \wedge dz = 2i dx \wedge dy$. If $f(z,\bar{z}) = f(z)$, we obtain the Cauchy's integral theorem

$$\int_{\partial C_2} f(z)dz = 0. \tag{8}$$

In fact, for $\varphi = f dz$, we have

$$d\varphi = df \wedge dz = d(u+iv) \wedge d(x+iy) = \left[-\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + i \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right] dx \wedge dy. \tag{9}$$

Namely, $d\varphi = 0$ is equivalent to the Cauchy-Riemann equations.

The generalized Cauchy's formula reads

$$f(z,\bar{z}) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta,\bar{\zeta})}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_D \frac{1}{\zeta - z} \frac{\partial f(\zeta,\bar{\zeta})}{\partial \bar{\zeta}} d\bar{\zeta} \wedge d\zeta. \tag{10}$$

If $f(z,\bar{z}) = f(z)$, this reduces to the Cauchy's integral formula. Verify the generalized Cauchy's formula for $f = z\bar{z}$ and a unit circle D

Poincare's lemma dictates that in a simply connected region, $d\varphi = 0 \Leftrightarrow \varphi = du$, which guarantees the presence of the potential. If $\boldsymbol{F}$ is a conserved force, $\mathrm{rot}\boldsymbol{F} = 0$, then $\boldsymbol{F} = \nabla\phi$ (see also below). This is generalized to multiply connected region (de Rham's theorem), and complex variables (Dolbeault's lemma).

EXAMPLES

Electromagnetic potential

If we set

$$\alpha = -c(E_x dx + E_y dy + E_z dz) \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \tag{11}$$

$$\begin{aligned} d\alpha &= dt \wedge [\{c(\partial_y E_z - \partial_z E_y) + \partial_t B_x\} dy \wedge dz + \{c(\partial_z E_x - \partial_x E_z) + \partial_t B_y\} dz \wedge dx + \{c(\partial_x E_y - \partial_y E_x) + \partial_t B_z\} dx \wedge dy] \\ &+ (\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz = 0 \end{aligned} \tag{12}$$

gives the Maxwell's equations

$$c\, \mathrm{rot}\mathbf{E} + \partial_t \mathbf{B} = 0, \mathrm{div}\mathbf{B} = 0. \tag{13}$$

Poincare's lemma guarantees the presence of the potential:

$$\alpha = d\beta, \beta = c\phi dt + A_x dx + A_y dy + A_z dz. \tag{14}$$

In fact, we have

$$\begin{aligned} d\beta &= [(c\partial_x\varphi - \partial_t A_x)dx + (c\partial_y\varphi - \partial_t A_y)dy + +(c\partial_z\varphi - \partial_t A_z)dz] \wedge dt \\ &+ (\partial_y A_z - \partial_z A_y) dy \wedge dz + (\partial_z A_x - \partial_x A_z) dz \wedge dx + (\partial_x A_y - \partial_y A_x) dx \wedge dy. \end{aligned} \tag{15}$$

First law of thermodynamics and internal energy

Let ω_Q and ω_L the heat absorbed and the work done by the system ($\omega_L = PdV$), respectively. The first law of thermodynamics dictates that for arbitrary closed path C in state space, one has

$$\oint_C \omega_Q = \oint_C \omega_L. \tag{16}$$

Poincare's lemma dictates that $\omega_Q - \omega_L$ is exact: there exists a function U such that

$$\omega_Q - \omega_L = dU. \tag{17}$$

U is called *internal energy.*

Hamiltonian

Consider a 2-form ω and a vector field X on a 2D manifold described by local coordinates q and p:

$$\omega = dp \wedge dq, \quad X = \frac{d}{dt} = \frac{dq}{dt}\frac{d}{dq} + \frac{dp}{dt}\frac{d}{dp}. \tag{18}$$

The Lie derivative of ω with respect to X is given by

$$L_X \omega = d\left\langle \omega, X \right\rangle + \left\langle d\omega, X \right\rangle = d\left\langle \omega, X \right\rangle. \tag{19}$$

If X is a Hamiltonian vector field, i.e., $L_X\omega = 0$, then Poincare's lemma dictates that there exists a function H which satisfies

$$\left\langle \omega, X \right\rangle = -dH. \tag{20}$$

Since

$$\left\langle \omega, X \right\rangle = \left\langle dp, X \right\rangle dq - \left\langle dq, X \right\rangle dp = \frac{dp}{dt}dq - \frac{dq}{dt}dp, \tag{21}$$

we obtain the equation of motions

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \tag{22}$$

THE FROBENIUS CONDITION

Consider a 1-form ω on $\mathbb{R}^n$. If $\omega \wedge d\omega = 0$, there exist functions f and g such that $\omega = fdg$. $\omega \wedge d\omega = 0$ is called the Frobenius condition which gives integrability condition. For example, for $\omega = Pdx + Qdy + Rdz$, we have the integrability condition:

$$\omega \wedge d\omega = \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \nabla \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = 0. \tag{23}$$

Now, consider a 1-form on a 2D manifold described by local coordinates V and U:

$$\omega = PdV + dU. \tag{24}$$

Here, P, V and U are the pressure, the volume, and the internal energy of a system, respectively. Since $\omega \wedge d\omega = 0$ is satisfied, there exist an integrating factor $1/T$ and a function S such that

$$dS = \frac{1}{T}\omega \Leftrightarrow TdS = PdV + dU. \tag{25}$$

S is called entropy.

CONFORMAL TRANSFORMATIONS AND CAUCHY-RIEMANN EQUATIONS

Consider a line element with a metric $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu} dr^\mu dr^\nu. \tag{26}$$

Under a transformation $\mathbf{r} \to \mathbf{r}'$, the metric changes as

$$g'_{\mu\nu}(\mathbf{r}') = \frac{\partial r^\alpha}{\partial r'^\mu} \frac{\partial r^\beta}{\partial r'^\nu} g_{\alpha\beta}(\mathbf{r}). \tag{27}$$

Conformal transformations are defined such that the angle between two vectors are kept invariant, which is described by a local scale factor in the metric:

$$g'_{\mu\nu}(\mathbf{r}') = \Omega(\mathbf{r}) g_{\alpha\beta}(\mathbf{r}) \tag{28}$$

since $\mathbf{r} \cdot \mathbf{r}' = g_{\mu\nu} r^\nu r'^\mu$. Illustrate some examples of conformal transformations. Now, consider an infinitesimal transformation of the form $r^\mu \to r^\mu + \varepsilon^\mu(\mathbf{r})$. Then, we have

$$ds^2 \to ds^2 + (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) dr^\mu dr^\nu. \tag{29}$$

Here, $\varepsilon_\mu = g_{\mu\nu}\varepsilon^\nu$. Conformal invariance requires

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = C g_{\mu\nu} \tag{30}$$

with some function C. By multiplying $g^{\mu\nu}$, we obtain $C = \frac{2}{d}\partial_\lambda \varepsilon^\lambda$.

Now, consider a two dimensional flat (Euclidean) space ($d = 2, g_{\mu\nu} = \delta_{\mu\nu}$). Then, we obtain the Cauchy-Riemann equations from the above equations:

$$\partial_x \varepsilon_x = \partial_y \varepsilon_y,\ \partial_x \varepsilon_y = -\partial_y \varepsilon_x \tag{31}$$

This is equivalent to $\frac{\partial}{\partial \bar{z}}\varepsilon = 0$, and hence we can express the transformation as

$$z \to z' = f(z). \tag{32}$$