Polar representation of a matrix

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Lemma. For any non-negative definite hermitian matrix H, there exists a hermitian matrix X which satisfies $H = \exp(X)$ and vice versa.

Proof. For $Hx_i = a_i x_i$, let $b_i = \log a_i$ where $a_i, b_i \in \mathbb{R}$. Then, define X as $Xx_i = b_i x_i$.

Theorem. Any non-singular matrix A can be represented as the product of an non-negative definite hermitian H with a unitary matrix U uniquely.

Proof. Let $H^2 = AA^\dagger$. Then, H is non-negative definite hermitian. For $H^2 = \exp(X)$, one may write $H = \exp(X/2)$. Now, we have $UU^\dagger = H^{-1}AA^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$. Therefore, we have the representation of A with the non-negative definite hermitian H and the unitary matrix U as

$$A = HU. \tag{1}$$

Note. If A is a normal matrix ($[A, A^\dagger] \neq 0$), we can also prove this as follows. For a unitary matrix U, one has

$$U^\dagger AU = \begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & ... \end{pmatrix} = \mathrm{diag}(\alpha_\mathrm{i}). \tag{2}$$

Now, decompose α_i as $\alpha_i = r_i e^{i\theta_i}$ where $r_i \geq 0$ and $\theta_i \in \mathbb{R}$. Then, one obtains $\mathrm{diag}(\alpha_i) = \mathrm{diag}(r_i)\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_\mathrm{i}})$, and hence $A = U\mathrm{diag}(r_i)U^\dagger U\mathrm{diag}(e^{i\theta_i})U^\dagger$. It is easy to show that $U\mathrm{diag}(r_i)U^\dagger$ and $U\mathrm{diag}(e^{i\theta_i})U^\dagger$ are non-negative definite hermitian and unitary, respectively.