# Fourier and Laplace transformations and complex analysis

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## FOURIER TRANSFORMATION

Fourier transformation is defined as

$$f(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ikx} f(x). \tag{1}$$

The inverse Fourier transformation is defined as

$$f(x) = \int_{-\infty}^{\infty} dx e^{ikx} f(k). \tag{2}$$

*Example.* Consider the diffusion equation

$$\left(\frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}\right) u(x,t) = \delta(x-\xi)\delta(t) \tag{3}$$

with $a > 0$ and $u(x,t) = 0$ for $t < 0$. With the Fourier transformation,

$$u(x,t) = \int_{-\infty}^{\infty} dk d\omega e^{ik(x-\xi)} e^{-i\omega t} \tilde{u}(k,\omega) \tag{4}$$

we obtain

$$\tilde{u}(k,\omega) = \frac{1}{4\pi^2}\frac{1}{-i\omega + ak^2}. \tag{5}$$

Thus,

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk d\omega e^{ik(x-\xi)} e^{-i\omega t}\frac{1}{-i\omega + ak^2} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-\xi)} e^{-ak^2 t} \\ &= \frac{1}{2\sqrt{\pi a t}}\exp\left(-\frac{(x-\xi)^2}{4at}\right) \equiv G(x-\xi,t). \end{aligned} \tag{6}$$

Now, consider the diffusion equation

$$\left(\frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}\right) u(x,t) = 0 \tag{7}$$

for $t > 0$ under the initial condition $u(x,0) = f(x)$. The solution is then given by

$$u(x,t) = \int_{-\infty}^{\infty} d\xi G(x-\xi,t) f(\xi). \tag{8}$$

## LAPLACE TRANSFORMATION

Laplace transformation is defined as

$$y_L(p) = \int_0^{\infty} dx e^{-px} y(x) = L[y(x)] \tag{9}$$

while its inverse transformation is defined as

$$L^{-1}[y_L(p)] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dp e^{px} y_L(p). \tag{10}$$

$\gamma$ is chosen so that all poles of $y_L(p)$ lie in the left side of the integration path.

*Example.* Consider the differential equation

$$\frac{d^2 y}{dx^2} + y = f(x) \tag{11}$$

for $x \geq 0$ with the initial condition $y(0) = y'(0) = 0$. Now, using

$$f_L(p) = \int_0^{\infty} dx e^{-px} f(x) \tag{12}$$

and

$$L[y''(x)] = p^2 y_L(p) - p y(0) - y'(0), \tag{13}$$

we have

$$y_L(p) = \frac{f_L(p)}{p^2+1}. \tag{14}$$

Therefore, we arrive at

$$y(x) = \int_0^x f(\xi) L^{-1}\left[\frac{1}{p^2+1}\right]_{x-\xi} d\xi \tag{15}$$

with the use of convolution theorem, where

$$L^{-1}\left[\frac{1}{p^2+1}\right] = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} dp e^{px}\frac{1}{p^2+1} = \sin x. \tag{16}$$

Finally, we get the solution

$$y(x) = \int_0^x f(\xi)\sin(x-\xi) d\xi. \tag{17}$$

*Exercise.* Charge accumulated at the capacitor $q(t)$ in the circuit obeys the following equation (Kirchhoff's laws). Solve for $q(t)$, using Laplace transformation,

$$L\frac{d^2 q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{q(t)}{C} = P(t) \tag{18}$$

under the initial condition $q(0) = Q$, $\frac{dq}{dt}(0) = I$.