Fluid mechanics, topology, and complex analysis

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COMPLEX POTENTIAL AND ITS FLOW

Regular functions may be visualized (or "plotted") by drawing their "flow". Namely, regular functions are equivalent to incompressible and irrotational fluids (or fields). This becomes clear by introducing complex potentials. The complex potential is defined as

$$f(z) = u(x, y) + iv(x, y) \tag{1}$$

where the "velocity" is given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = v_x$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = v_y$. Then, one can show

$$f'(z) = v_x - iv_y, \quad du = v_s ds, \quad dv = v_n ds \tag{2}$$

where v_s and v_n are components of the velocity tangent and normal to the line element ds, respectively. It can be seen that $u = const.$ and $v = const.$ correspond to equipotential curve and stream line, respectively. Hence, u and v are dubbed potential and stream function, respectively. Illustrate flows of $f(z) = Az^n (n = \pm1, \pm2, \pm3), m \log z, i\kappa \log z, Uz + m \log \frac{z+a}{z-a}, U\left(z + \frac{a^2}{z}\right), U\left(z + \frac{a^2}{z}\right) + i\kappa \log z$ (see Fig. 1).

Now, consider the complex potential

$$f(z) = U\left(z + \frac{a^2}{z}\right). \tag{3}$$

Since

$$\mathrm{Im} f = U \sin\theta \left(r - \frac{a^2}{r}\right), \tag{4}$$

$r = a$ is one of the stream lines. Thus, this potential provides a flow along a cylinder with radius a. This potential may be obtained in three different ways.

1. The Joukowski transformation.

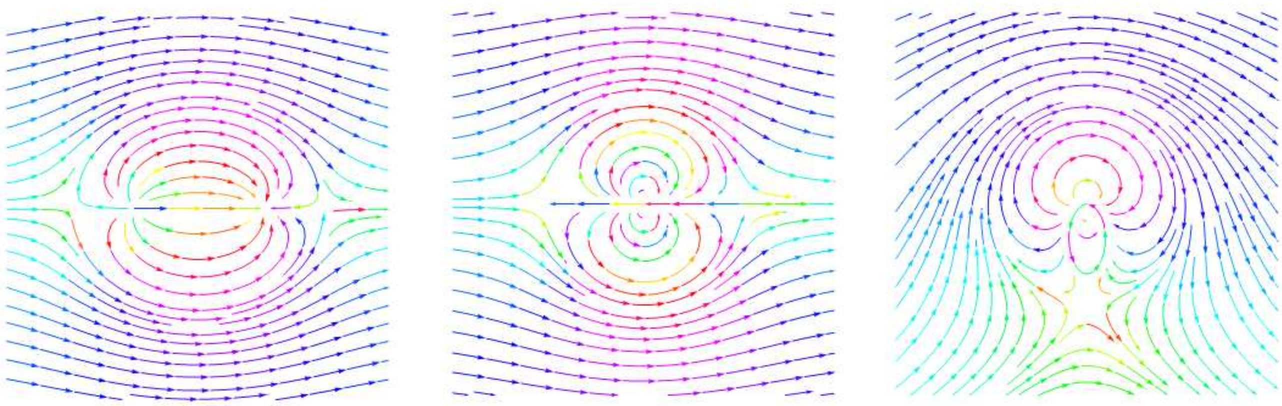

FIG. 1: Flows of $f(z) = Uz + m \log \frac{z+a}{z-a}, U\left(z + \frac{a^2}{z}\right)$ and $U\left(z + \frac{a^2}{z}\right) + i\kappa \log z$ (from left to right).

Let us use the Joukowski transformation

$$z = \frac{a}{2b}\left(\zeta + \frac{b^2}{\zeta}\right). \tag{5}$$

With this conformal mapping, the cylinder $|\zeta| = b$ is mapped onto a slab $-a < x < a$. Thus, a uniform flow along x-axis $f(z) = U'z$ in z-plane describes a flow along the cylinder in ζ-plane:

$$f(z) = U'z = U\left(\zeta + \frac{b^2}{\zeta}\right) \tag{6}$$

with $U = \frac{a}{2b}U'$.

2. Principle of mirror image about a circle (or Milne-Thomson circle theorem).

If $f(z)$ is regular on a region D and continuous on D and an arc Γ, where Γ is a part of ∂D, and $\mathrm{Im}f(z) = b$ on Γ, then $f(z)$ can be analytically-continued on a mirror region D^* with respect to Γ as

$$f^*(z) = \bar{f}\left(\bar{z}_0 + \frac{R^2}{z - z_0}\right) + 2bi \tag{7}$$

where z_0, R and b are the center of the circle, the radius of the circle, and an arbitrary real constant, respectively.

Now, considering $f(z) \to Uz$ at $z \to \infty$ and taking $\mathrm{Im}f = 0$ on Γ, we have $f^*(z) \to U\frac{a^2}{z}$ at $z \to 0$ and thus finally arrive at

$$f(z) = U\left(z + \frac{a^2}{z}\right). \tag{8}$$

3. Method of conjugate harmonic function.

Let us expand a regular function $f(z)$ around the origin within a cylinder

$$f(z) = \sum_{n=0} c_n z^n, \quad |z| < a. \tag{9}$$

For $c_n = a_n + ib_n$, we have

$$u(re^{i\theta}) = a_0 + \sum_{n=1} (a_n \cos n\theta - b_n \sin n\theta) r^n, \tag{10}$$

$$v(re^{i\theta}) = b_0 + \sum_{n=1} (a_n \sin n\theta + b_n \cos n\theta) r^n. \tag{11}$$

Now, define $R(\theta) = u(ae^{i\theta})$ and $I(\theta) = v(ae^{i\theta})$. The conjugate Fourier series is defined as

$$R^*(\theta) = \sum_{n=1} (a_n \sin n\theta + b_n \cos n\theta) a^n. \tag{12}$$

Then, we obtain

$$R(\theta) = a_0 - I^*(\theta), \quad I(\theta) = b_0 + R^*(\theta). \tag{13}$$

Similarly, we expand a regular function $f(z)$ outside a cylinder as

$$f(z) = \sum_{n=0} c_n z^{-n}, \quad |z| > a \tag{14}$$

and obtain the following relations in a similar fashion

$$R(\theta) = a_0 + I^*(\theta), \quad I(\theta) = b_0 - R^*(\theta). \tag{15}$$

Now, let us consider a cylinder under the complex potential $f_0(z)$ which is regular within the cylinder ($|z| \leq a$). The presence of the cylinder modifies the complex potential as $f(z) = f_0(z) + f_1(z)$ where $f_1(z)$ is regular outside the cylinder $|z| \geq a$. Correspondingly, on the cylinder, we have

$$u(\theta) = u_0(\theta) + u_1(\theta), \quad v(\theta) = v_0(\theta) + v_1(\theta). \tag{16}$$

Since $|z| = a$ is the stream line, we can set $v(\theta) = 0$ on $|z| = a$. Then, using the above results, we have

$$v_1(\theta) = -v_0(\theta) => u_1(\theta) - \mathrm{Re}f_1(\infty) = u_0(\theta) - \mathrm{Re}f_0(0). \tag{17}$$

Therefore, we obtain

$$u(\theta) = 2u_0(\theta). \tag{18}$$

Here, we have dropped unimportant constant. This indicates that the velocity of a fluid on the cylinder is twice as large as that without the cylinder. For a uniform flow $f_0(z) = Uz$, we have

$$u(\theta) = 2Ua\cos\theta. \tag{19}$$

LIFT FORCE

Now, we will investigate a force acting on the cylinder. First, consider the complex potential

$$f(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\log z \tag{20}$$

which describes a flow around a cylinder $|z| = a$ with a circulation Γ. For an incompressible fluid (constant density), the Bernoulli theorem states

$$p + \frac{\rho_0}{2}v^2 = const. = p_0 \tag{21}$$

where p is the pressure and ρ_0 is the density. Using this theorem, we can calculate the force (F_x and F_y) acting on the cylinder as

$$F_x + iF_y = \int_C p(x,y)idz = -i\frac{\rho_0}{2}\int_C |f'(z)|^2\,dz = -i\frac{\rho_0}{2}2\pi i \cdot i\frac{\Gamma}{\pi}U = i\rho_0\Gamma U. \tag{22}$$

Note that along the stream line, we have $df = d\bar{f}$. Finally, we obtain

$$F_x = 0, \quad F_y = \rho_0\Gamma U. \tag{23}$$

This is an example of the Kutta-Joukowski's formula of lift force where the above relation is proven for any complex potential which gives a flow around an obstacle with a circulation. Physically, the circulation gives asymmetry of the flow on upper and lower sides of the cylinder and hence it receives the force perpendicular to the uniform flow (see Fig. 1).

Exercise. Obtain the stagnation points $v_x = v_y = 0$ (or $f'(z) = 0$) of the above complex potential Eq.(20).

APPLICATION TO OPTICS

Here, let us show how conformal mapping can be used to describe "invisibility". Consider a two-dimensional medium characterized by an inhomogeneous refractive-index $n(x, y)$. Then, the amplitudes ψ of the two polarizations of light obey the wave (Helmholtz) equation,

$$\left(\nabla^2 + \frac{\omega^2}{v^2}\right)\psi = 0 \tag{24}$$

where ω and $v = c/n$ are the frequency and the velocity of light in the medium. c is the speed of light in vacuum. With the use of complex coordinate z, the wave equation is reduced to

$$\left(4\frac{\partial^2}{\partial z\partial\bar{z}} + n^2\frac{\omega^2}{c^2}\right)\psi = 0. \tag{25}$$

Now, consider conformal mapping $z \to w$ where $w(z)$ is an analytic function of z. Then, the equation is transformed into

$$\left(4\frac{\partial^2}{\partial w \partial \bar{w}} + (n')^2 \frac{\omega^2}{c^2}\right)\psi = 0 \tag{26}$$

in w space with

$$n = n' \left|\frac{dw}{dz}\right|. \tag{27}$$

If we set

$$n = \left|\frac{dw}{dz}\right|, \tag{28}$$

we have $n' = 1$. Namely, in w coordinates, the wave propagates in "vacuum" where light rays propagate along straight lines. If $w(z) \to z$ at $z \to \infty$, all incident waves go to infinity as if they have come from vacuum in z space as long as they avoid branch cuts. For example, consider the mapping

$$w(z) = z + \frac{a^2}{z}, \quad z = \frac{1}{2}\left(w \pm \sqrt{w^2 - 4a^2}\right). \tag{29}$$

If light rays pass the branch cut $-2a \leq w \leq 2a$, they will go into the branch $z = \frac{1}{2}\left(w - \sqrt{w^2 - 4a^2}\right)$ and reach $z = 0$ at $w \to \infty$, namely they are after all absorbed at the origin in z space. Therefore, from some directions, anything near the origin is invisible while from other directions, it is visible, also as anticipated from Fig.1.

By adding an appropriate potential, one can make up almost invisible devices. See, for details, U. Leonhardt, Science 312, 1777 (2006); New Journal of Physics 8, 118 (2006).

POINCARÉ-HOPF THEOREM

In the above, we have seen several examples of flows described by complex functions. Some vector fields (or flows) have singularities where they are ill-defined. The number of singularities of a vector field is profoundly related to the topology of the system on which the vector field is attached. To show this, let us first define the index $\gamma_p(X)$ for a vector field X and a singular point p of X surrounded by a closed path C (see Figs. 2 and 3) as

$$\gamma_p(X) = \frac{1}{2\pi}\int_C d\theta. \tag{30}$$

The Poincaré-Hopf theorem states that the sum of the indices over the closed surface S is equal to the Euler characteristic of S:

$$\sum_p \gamma_p(X) = \chi(S). \tag{31}$$

The Euler characteristic is a global quantity and given by

$$\chi(S) = 2(1 - g) \tag{32}$$

where a genus g counts a number of the holes of S. For example, $\chi(S^2) = 2$ and $\chi(T^2) = 0$ (see Fig. 4). This means that on a torus T^2, there may be a vector field without any singular points. By applying this formula to S^2 and considering a flow emanating from the south pole and into the north pole, one can deduce the Euler's formula $F - E + V = \chi(S^2) = 2$ by dividing S^2 into F polygons. Here, E and V are the total number of edges and vertices, respectively.

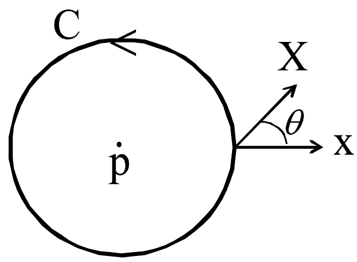

FIG. 2: Definition of θ.

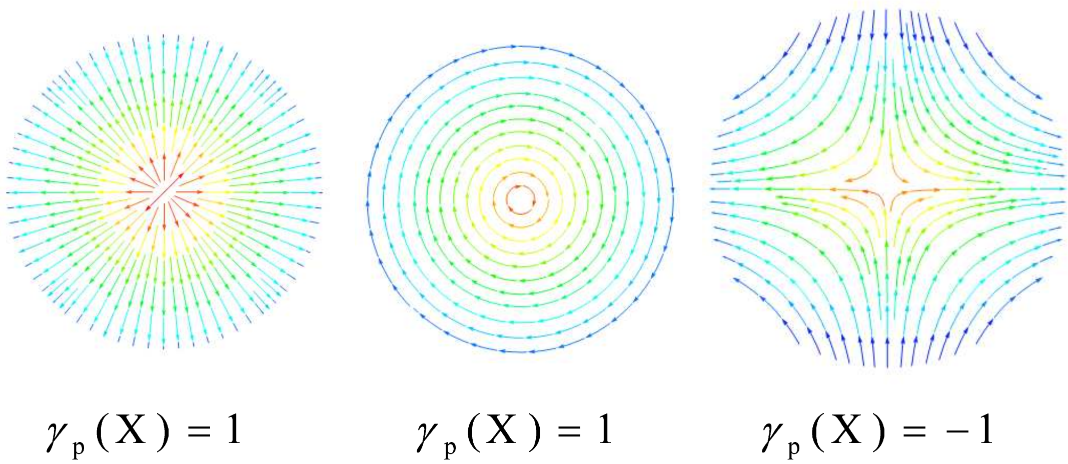

FIG. 3: Some examples of the index. The corresponding complex functions are $f(z) = \log z, -i\log z, z^2$.

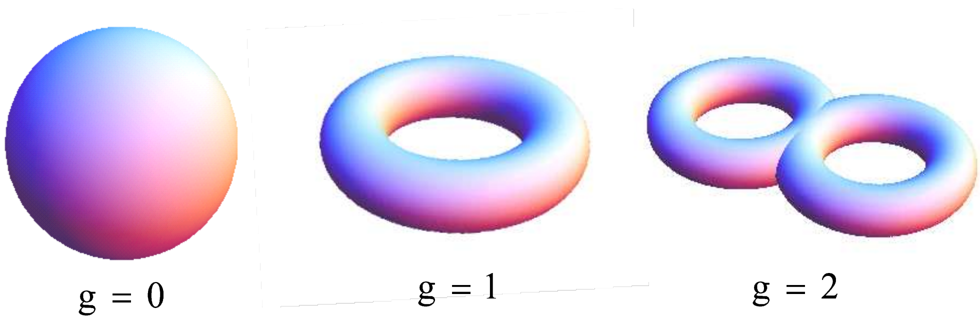

FIG. 4: Some examples of the Euler characteristics.