Residue at infinity

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It is useful to introduce the residue at infinity. It is defined as

$$\mathrm{Res}(f(z);\infty) \equiv -\frac{1}{2\pi i}\oint_C f(z)dz \tag{1}$$

where C is a simple closed path in $R < |z| < \infty$. If we expand $f(z)$ as $f(z) = \sum a_n z^n$ in $R < |z| < \infty$, we have $\mathrm{Res}(f(z);\infty) = -a_{-1}$. By changing the variable as $w = 1/z$, the residue can be expressed as

$$\mathrm{Res}(f(z);\infty) = -\frac{1}{2\pi i}\oint_{C'} f(1/w)\frac{dw}{w^2} = -\,\mathrm{Res}(f(1/z)/z^2;0) \tag{2}$$

where C' is a simple closed path in $0 < |w| < 1/R$. If $f(\infty) = 0$, we can write $\mathrm{Res}(f(z);\infty) = -\lim_{z\to\infty} zf(z)$.

Let z_i denote the finite singularities of the (rational) function $f(z)$. Then, we can derive the following sum rule:

$$\sum_i \mathrm{Res}(f(z);z_i) + \mathrm{Res}(f(z);\infty) = 0. \tag{3}$$

Thus, when $\lim_{z\to\infty} zf(z) = 0$, the sum rule is reduced to $\sum_i \mathrm{Res}(f(z);z_i) = 0$.

Let us illustrate some examples. $\mathrm{Res}(1/z;\infty) = -1$. Note that even if $f(z)$ is analytic at infinity, it has a residue there.

Since

$$\frac{a^2 - z^2}{a^2 + z^2}\frac{1}{z} = -\frac{1}{z} + ... \tag{4}$$

at $z \to \infty$, for a simple closed contour C enclosing the points $z = 0, \pm ia$, we obtain

$$\frac{1}{2\pi i}\oint_C \frac{a^2 - z^2}{a^2 + z^2}\frac{dz}{z} = -1. \tag{5}$$

Confirm the sum rule in the above examples.