Definite integrals

Takehito Yokoyama
Department of Physics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan
(Dated: April 15, 2019)

Let z_i be the poles of a function $f(z)$ on some regions. Under appropriate conditions, definite integrals may be calculated as follows

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_i \mathrm{Res}\left(f(z), z_i\right), \tag{1}$$

$$\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta = \int_{|z|=1} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} = 2\pi \sum_i \mathrm{Res}\left(f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{z}, z_i\right), \tag{2}$$

$$\int_0^{\infty} x^{\alpha} f(x)dx = \frac{2\pi i}{1-e^{2\alpha\pi i}} \sum_i \mathrm{Res}\left(z^{\alpha} f(z), z_i\right), \tag{3}$$

$$\int_0^{\infty} f(x)\log x dx = \pi^2 \mathrm{Res}\left(f(z), 1\right) - \frac{1}{2}\mathrm{Re}\sum_i \mathrm{Res}\left(f(z)\log^2 z, z_i\right), \tag{4}$$

$$\int_0^{\infty} f(x)\log x dx = \pi^2 \mathrm{Res}\left(f(z), 1\right) + \mathrm{Re}\sum_i \mathrm{Res}\left(f(z)\left(\pi i \log z + \frac{\pi^2}{2}\right), z_i\right), \ \mathrm{if} f(x) = f(-x), \tag{5}$$

$$\int_0^{\infty} f(x)dx = -\sum_i \mathrm{Res}\left(f(z)\log z, z_i\right), \tag{6}$$

$$\int_a^b f(x)dx = \sum_i \mathrm{Res}\left(f(z)\mathrm{Log}\left(\frac{z-b}{z-a}\right), z_i\right), \tag{7}$$

$$\int_{-\infty}^{\infty} f(x)dx = \frac{2\pi i}{1-g(a)} \sum_i \mathrm{Res}\left(f(z), z_i\right), \ \mathrm{if} f(x+ia) = f(x)g(a), \tag{8}$$

$$\int_0^{\infty} f(x)dx = \frac{2\pi i}{1-g(\theta)e^{i\theta}} \sum_i \mathrm{Res}\left(f(z), z_i\right), \ \mathrm{if} f(ze^{i\theta}) = f(z)g(\theta). \tag{9}$$

The integrals can be calculated by adding some contours over which the integrals vanish in some limits, become proportional to the original integrals by some variable transformations, or can be calculated directly (such as Fresnel integrals).